

A MODIFIED METHOD FOR BOOTSTRAP CONFIDENCE INTERVALS

By

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To my wife
and
to the memory of my mother

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The bootstrap method was introduced as a resampling method for error estimation. It has later become a general purpose method applicable to a wide range of statistical problems.

Confidence intervals constructed by the bootstrap method have been suggested in the statistical literature largely for the purposes of defining refined bootstrap confidence intervals and of obtaining high-order asymptotic properties. However, confidence intervals constructed by bootstrap procedures are known to be unsatisfactory for small samples. This is not surprising because there seems no way for the bootstrap method to recognize the true population distribution from a small sample.

The purpose of this research was to identify a scheme to reduce the coverage probability discrepancy of existing bootstrap confidence intervals in small samples.

Given a family of distributions $\mathcal{F}$, the coverage probability of confidence interval for the mean was found to be asymptotically correlated with the Kolmogorov-Smirnov distances between the sample empirical distributions and a preferred distribution. The preferred distribution is defined as the distribution that has the coverage probability closest to the nominal level of confidence interval among all the distributions in

$\mathcal{F}$. Based on this information, the Kolmogorov-Smirnov distances between the sample empirical distribution and the preferred distribution were used to modify confidence intervals. This modified method produced improvements in coverage accuracy in two-sided intervals for the mean. The method was then applied for the variance, the trimmed mean, the correlation coefficient, and the testing problem of separate families of distributions. Through a simulation study, the modified intervals are shown to be better than the existing bootstrap confidence intervals in the literature.

CHAPTER 1 INTRODUCTION

In parameter estimation, the parameter we wish to estimate usually has an intuitive meaning and can be estimated by an intuitive formula from the data, e.g., the sample mean for the population mean and the sample variance for the population variance. No assumption on the population distribution is necessary at the estimation stage. But finding the reliability of the estimates is no longer intuitive. For example, there is no intuitive method to find the relation between α and δ in the confidence interval equation

$$Pr\{|\hat{\theta} - \theta| < \delta\} = 1 - 2\alpha, \quad (1.1)$$

where θ and $\hat{\theta}$ are, respectively, the target and estimated parameters. A large portion of classic statistical theory was directed toward quantifying this relation. However, if the population distribution is known, then (1.1) can be easily established using computer simulation. Let the sample size be n . Then n observations can be simulated from the known population distribution, and the estimate $\hat{\theta}$ of θ can be obtained from the simulated sample. If $\hat{\theta}$ is simulated a large number of times, its distribution can be estimated with great accuracy. Consequently, (1.1) is obtained. The main difficulty of this approach is, of course, that the population distribution is unknown. Efron(1979) suggested that one might use the sample (empirical) distribution as the true population. This is known as the bootstrap method. It is very intuitive because the sample distribution provides the best information we have for the population

distribution. Unless one has some prior knowledge of the population distribution, there seems to be no better way to represent it.

Historically, after introduction of the bootstrap method by Efron(1979), fundamental work (e.g., Bickel and Freedman, 1981; Singh, 1981) established the asymptotic consistency of the bootstrap. Then, several confidence intervals constructed by bootstrap methods were suggested largely for the purposes of defining refined bootstrap confidence intervals and of obtaining high-order asymptotic properties (e.g., Efron, 1987; Beran, 1987; Martin, 1990). It is difficult to choose a single confidence interval as the best of all the confidence intervals. Hall(1988, 1992) developed a unified theoretical framework within which different bootstrap critical points may be compared and evaluated, based on asymptotic expansion methods.

It has been shown that under very general conditions, the bootstrap method is an asymptotically valid method for obtaining the confidence intervals (Bickel and Freedman, 1981; Singh, 1981; Efron, 1987). Again, this is not surprising because when the sample size is large, the empirical distribution is almost the same as the true population distribution. The problem is with a small sample. Few theoretical investigations have been done in this area due to the analytic complexity of the bootstrap method. Simulation has been the main tool. Although there were favorable reports on small sample bootstrap confidence intervals (Diciccio and Romano, 1988; Efron, 1987), there were also reports which cast doubt on their accuracy (Noreen, 1989; Schenker, 1985; Loh and Wu, 1987). Many reports also dealt with the problem of poor coverage probabilities of confidence intervals (Loh, 1987; Beran, 1987, 1988; Hall and Martin, 1988; Martin, 1990). This dissertation will try to answer a fundamental question of small sample bootstrapping. How does bootstrap recognize the population distribution from a small sample? Since it is almost impossible to differentiate most distributions with a small sample by statistical testing, we wonder how can the bootstrap method treat them differently in confidence interval construction.

If it can not, then what is the preferred distribution that the bootstrap procedure is based on?

In answering these questions, a new way to improve the existing confidence interval construction methods was discovered based on the Kolmogorov-Smirnov distance between the sample and a preferred distribution. In most cases, the improvements turn out to be substantial.

This dissertation will be divided into five chapters as described below. Chapter 2 describes currently used bootstrap methods for constructing confidence intervals and a review of the current literature. Chapter 3 is devoted to developing a modified confidence interval method. The results of a simulation study are presented in Chapter 4. The two-sided confidence intervals for the mean, the variance, the trimmed mean, and the correlation coefficient are considered in this chapter. As another application of the modified method, the testing problem of separate families of distributions is considered. A comparison of the modified method and the existing bootstrap confidence interval methods is given for each case. Finally, Chapter 5 summarizes the results and presents a brief discussion of future research issues.

CHAPTER 2 EXISTING BOOTSTRAP CONFIDENCE INTERVAL CONSTRUCTION METHODS

2.1 Literature Review

The bootstrap method, introduced by Efron(1979), was originally introduced as a resampling method to reduce bias and to provide more reliable standard error of $\hat{\theta}$, where $\hat{\theta}$ is an estimate of a parameter of interest θ . For example, suppose we have a random sample $\mathbf{x} = (x_1, \dots, x_n)$ from an unknown distribution F which is associated with a parameter θ . Suppose we are interested in estimating θ , and have selected an estimator $\hat{\theta}$. Moreover, we need to assess the accuracy of $\hat{\theta}$. The common measure of accuracy for $\hat{\theta}$ is the standard error. The problem is that it is usually not easy to estimate the standard error of $\hat{\theta}$ except when $\hat{\theta}$ is the sample mean.

There was an earlier resampling method for estimating the standard error of $\hat{\theta}$. Using Quenouille's (1956) technique for estimating biases, Tukey(1958) introduced the jackknife method for error estimation.

Let $\hat{\theta}$ be an estimator of θ based on $\mathbf{x}$. Let $\hat{\theta}_{(i)}$ be the estimator of θ using $\mathbf{x}_{(i)} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. The jackknife estimate of variance is given by

$$\begin{aligned}\hat{\sigma}_J^2 &= \frac{n-1}{n} \sum_{i=1}^n \left(\hat{\theta}_{(i)} - \frac{1}{n} \sum_{j=1}^n \hat{\theta}_{(i)} \right)^2 \\ &= \frac{1}{n(n-1)} \sum_{i=1}^n (p_i - \bar{p})^2,\end{aligned}\tag{2.1}$$

where $p_i = n\hat{\theta} - (n-1)\hat{\theta}_{(i)}$ and $\bar{p} = \sum_{j=1}^n p_j/n$. If θ is the mean of a population, equation (2.1) reduces to the formula for the variance of the sample mean. The jackknife can produce an estimate of standard error for most complicated estimator $\hat{\theta}$. However, if $\hat{\theta}$ is an unsmooth statistic like the median, the jackknife estimate of standard error of $\hat{\theta}$ is inconsistent. Also, for nonlinear statistic like the correlation coefficient, the jackknife estimate can be inefficient (Efron and Tibshirani, 1993).

While the jackknife resamples systematically, the bootstrap resamples at random. Also, by repeating this resampling procedure, the bootstrap approximates the sampling distribution of $\hat{\theta}$ under F by the sampling distribution of $\hat{\theta}$ under $\hat{F}_n$, where $\hat{F}_n$ is the sample cumulative distribution function(cdf) which assigns mass $1/n$ to each x_i . Bootstrap suggests the following Monte Carlo algorithm (Efron, 1981, 1982) for standard error estimation.

1. Construct $\hat{F}_n$.
2. Draw a bootstrap sample $x_1^*, \dots, x_n^*$ iid with cdf $\hat{F}_n$ and calculate $\hat{\theta}^* = \hat{\theta}(x_1^*, \dots, x_n^*)$.
3. Independently repeat step 2 a large number of times, obtaining $\hat{\theta}_1^*, \dots, \hat{\theta}_B^*$,

where B is the number of replicates.

Then, the bootstrap estimate of variance is given by

$$\hat{\sigma}_B^2 = \frac{1}{B-1} \sum_{b=1}^B (\hat{\theta}_b^* - \bar{\theta}^*)^2, \quad (2.2)$$

where $\bar{\theta}^* = \sum_{b=1}^B \hat{\theta}_b^* / B$.

The basic idea, however, can be applied to a wide range of statistical problems. The bootstrap method above is sometimes called the nonparametric bootstrap. For the parametric bootstrap case, suppose that the sampling distribution F_θ is completely determined by an unknown parameter θ . First use the sample $(x_1, \dots, x_n)$ to estimate θ by the maximum likelihood principle or some other efficient estimate $\hat{\theta}$, and then use $F_{\hat{\theta}}$ instead of $\hat{F}_n$ in the Monte Carlo algorithm above (Efron, 1987). Bickel and Freedman(1981) and Singh(1981) showed the asymptotic consistency of

bootstrap estimators for sample mean and other statistics under some regularity conditions.

There are many papers on bootstrap confidence intervals. In a series of papers, Efron(1979, 1981, 1982) developed two methods, percentile method and bias-corrected (BC) percentile method, for constructing approximate bootstrap confidence interval for a real valued parameter θ . The BC method was developed to improve the performance of the percentile method when the median unbiasedness fails, i.e., $C\hat{D}F(\hat{\theta}) \neq 0.5$, where $C\hat{D}F(\hat{\theta})$ is the cdf of the bootstrap distribution of $\hat{\theta}^*$. However, Schenker(1985) showed that the coverage probabilities of the percentile and BC intervals were substantially below the nominal level in relatively simple problems such as estimating the variance of the normal distribution. Also, Diccio and Romano(1988) show unsatisfactory small-sample performances of these two methods in constructing confidence interval for the exponential mean.

Efron(1987) presented the accelerated bias-corrected percentile (BC_a) method. The difference between the BC method and the BC_a method was the underlying assumptions.

Let z_α be the 100α percentile point of a standard normal variate and $\hat{\sigma}$ be an estimate of the standard deviation of $\hat{\theta}$. A standard confidence interval of the form

$$(\hat{\theta} + z_\alpha \hat{\sigma}, \hat{\theta} + z_{1-\alpha} \hat{\sigma})$$

assumes

$$(\hat{\theta} - \theta)/\hat{\sigma} \sim N(0, 1). \quad (2.3)$$

The BC interval assumes that there exists some monotone-increasing transformation g such that the transformed quantities $\hat{\phi} = g(\hat{\theta})$ and $\phi = g(\theta)$ satisfy

$$(\hat{\phi} - \phi)/\tau \sim N(-b_0, 1), \quad (2.4)$$

where τ being the constant standard error of $\hat{\phi}$. The BC_a interval is based on a more general assumption: for some monotone-increasing transformation g , the transformed

quantities $\hat{\phi} = g(\hat{\theta})$ and $\phi = g(\theta)$ satisfy

$$(\hat{\phi} - \phi)/\tau \sim N(-b_0\sigma_\phi, \sigma_\phi^2), \quad \sigma_\phi = 1 + a\phi, \quad (2.5)$$

where b_0 and a are constants. In (2.3), it is assumed that the standardized statistic is normal with no bias and with unit variance. In (2.4), the normalizing and variance stabilizing transformation g is assumed, while in (2.5) only the normalizing transformation g is assumed. In other words, (2.5) assumes that the standardized statistic is normal with some bias and possibly with a standard deviation changing linearly with the parameter ϕ .

Efron(1987) showed that the BC_a method produced confidence interval for θ automatically without requiring any knowledge of the transformation g to form (2.5) and is correct to $O(n^{-1})$ for one-sided α -level confidence interval. This is not true for the standard interval, percentile interval and BC interval. They have true coverage probabilities which differ from α by $O(n^{-1/2})$. Diccio and Tibshirani(1987) present BC_a^0 method which is asymptotically equivalent to the BC_a interval and can be computed without bootstrap sampling.

Singh(1981) showed that for constructing confidence interval for the mean, the use of pivot $\sqrt{n}(\bar{x}_n - \mu)/\sigma$ in bootstrapping produced a coverage probability as good as one-term Edgeworth expansion.

Hall(1986) and Beran(1987) considered the issue of coverage probability of confidence interval from a different point of view. They utilized the bootstrap distribution of the studentized statistic $\sqrt{n}(\hat{\theta} - \theta)/s$, where s^2 is a consistent estimate of the variance of $\sqrt{n}(\hat{\theta} - \theta)$, and the iterated bootstrap to yield an approximation with an error of arbitrary small order. To reduce a coverage error, Loh(1987) proposed a calibrated interval which utilizes iterated bootstrap.

Beran(1987) presented the B-method which was based on the concept of prepivot-ing. Suppose a function $R(\theta)$ has been chosen to construct the confidence interval for

θ . He called this function the root of the confidence set. Prepivoting is the transformation of $R(\theta)$ by its estimated bootstrap cdf $\hat{W}(R(\theta))$ which limit law is uniform(0,1). Then, he treated the prepivoted statistic itself as a root $R_1(\theta) \equiv \hat{W}(R(\theta))$ for confidence sets concerning θ and derive an estimated quantile of its sampling distribution by the percentile method. The idea behind prepivoting is that the distribution of $R_1(\theta)$ is less dependent on F than the distribution of $R(\theta)$ whose limiting distribution may depend on F . The prepivoting can be iterated and Beran(1990) showed that the coverage error was reduced by further iteration. Martin(1990) showed that with iterated bootstrap procedures, bootstrap coverage correction produced improvements in coverage accuracy of order $n^{-1/2}$ in one-sided intervals, and of order n^{-1} in two-sided intervals.

It is difficult to choose a single bootstrap confidence interval as the best of all, because of different underlying assumptions. Hall(1988) developed a unified theoretical framework within which different bootstrap critical points might be compared and evaluated, based on asymptotic expansion methods.

2.2 Bootstrap Confidence Interval Methods

Let $\mathbf{x}=(x_1, x_2, \dots, x_n)$, $F(x)$, θ , and $\hat{\theta}$ be defined as before. Let $\hat{\theta}$ be an estimate of θ based on $\mathbf{x}$. Let $\hat{F}_n(x)$ be the empirical distribution defined by

$$\hat{F}_n(x) = \#\{x_i \leq x\}/n, \quad (2.6)$$

where $\#\{ \}$ means the number of times that the statement inside $\{ \}$ is true. Let $\mathbf{x}_b^* = (x_{b1}^*, x_{b2}^*, \dots, x_{bn}^*)$ be the b th bootstrap sample from (2.6), i.e., $\mathbf{x}_b^*$ is a sample of size n from $\mathbf{x}$ with replacement. Let $\hat{\theta}_b^*$ be the estimate of θ from $\mathbf{x}_b^*$, and let the total bootstrap replicates be B . A natural method to estimate the distribution of $\hat{\theta}$ is the

empirical bootstrap distribution

$$\hat{G}(y) = \#\{\hat{\theta}_b^* \leq y\}/B.$$

Consequently, a $100(1-2\alpha)\%$ confidence interval for θ can be approximated by

$$[\hat{G}^{-1}(\alpha), \hat{G}^{-1}(1-\alpha)]. \quad (2.7)$$

This is the percentile method. A simple modification of (2.7) is to make it centralized at $\hat{\theta}$, i.e., let the distribution $\hat{G}(y)$ be modified as

$$\hat{H}(y) = \#\{\hat{\theta}_b^* - \hat{\theta} \leq y\}/B, \quad (2.8)$$

and the $100(1-2\alpha)\%$ confidence interval for θ be

$$[\hat{\theta} - \hat{H}^{-1}(1-\alpha), \hat{\theta} - \hat{H}^{-1}(\alpha)], \quad (2.9)$$

provided that θ can take negative values. When θ is known to be a positive quantity, such as the variance, (2.8) and (2.9) are modified by

$$\hat{K}(y) = \#\{\hat{\theta}_b^*/\hat{\theta} \leq y\}/B,$$

and the $100(1-2\alpha)\%$ confidence interval becomes

$$[\hat{\theta}/\hat{K}^{-1}(1-\alpha), \hat{\theta}/\hat{K}^{-1}(\alpha)]. \quad (2.10)$$

Equations (2.9) and (2.10) are referred to as the centralized percentile confidence interval for the mean and the variance, respectively.

Another simple method is the percentile-t method (Hall, 1992) which takes the advantage of the robustness of a t-distribution confidence interval. The idea is to approximate the distribution of

$$t = \sqrt{n}(\hat{\theta} - \theta)/\tau \quad (2.11)$$

by the bootstrap distribution of

$$t_b = \sqrt{n}(\hat{\theta}_b - \hat{\theta})/\tau_b$$

where τ is a $\sqrt{n}$ consistent estimate of standard deviation of $\sqrt{n}(\hat{\theta} - \theta)$.

For $\theta = \mu$, approximate the distribution of

$$t = \sqrt{n}(\bar{x} - \mu)/s \quad (2.12)$$

by the bootstrap distribution of

$$t_b = \sqrt{n}(\bar{x}_b - \bar{x})/s_b \quad (2.13)$$

where μ , $\bar{x}$, $\bar{x}_b$ are respectively the true mean, sample mean, and the bootstrap sample mean, s and s_b are the sample and bootstrap standard deviations. Let $t_{b,\alpha}$ and $t_{b,1-\alpha}$ be the lower and upper α quantiles from the bootstrap sample distribution (2.13). Then, a $100(1-2\alpha)\%$ for μ becomes

$$[\bar{x} - t_{b,1-\alpha}s/\sqrt{n}, \bar{x} - t_{b,\alpha}s/\sqrt{n}]. \quad (2.14)$$

The percentile-t confidence interval for the variance follows the same idea. They are obtained by replacing equations (2.12 - 2.14), respectively, by

$$t = \sqrt{n}(s^2 - \sigma^2)/\tau, \quad \tau = \left\{ \sum_{i=1}^n (x_i - \bar{x})^4 / n - (n-3)s^4 / (n-1) \right\}^{1/2}$$

$$t_b = \sqrt{n}(s_b^2 - s^2)/\tau_b$$

and

$$[s^2 - t_{b,1-\alpha}\tau/\sqrt{n}, s^2 - t_{b,\alpha}\tau/\sqrt{n}],$$

where σ^2 is the true variance, τ^2/n is the estimate of the variance of s^2 (Mood et al., 1974), and τ_b is the τ from a bootstrap sample.

For more complex method, Efron(1982) suggested a bias-corrected percentile (BC) method by letting the confidence interval be

$$[\hat{G}^{-1}(\Phi(2b_0 + z_\alpha)), \hat{G}^{-1}(\Phi(2b_0 + z_{1-\alpha}))], \quad (2.15)$$

where $\Phi(\cdot)$ is the standard normal cdf, z_α denotes the 100α percentile point of a standard normal variate and the bias constant is defined as

$$b_0 = \Phi^{-1}(\hat{G}(\hat{\theta})).$$

Efron(1987) suggested another similar method, called accelerated bias-corrected percentile (BC_a) method. He first considered the simplest type of parametric problem, i.e., the data consists only of a single real-valued statistic $\hat{\theta}$ in a one-parameter family of densities $f_{\theta}(\hat{\theta})$. In fact, there is no need to use the bootstrap for constructing a confidence interval in this case, since it is usually easy to find a exact confidence interval for θ . The reasons for setting this case are to compare the parametric bootstrap confidence interval with a possible exact confidence interval and then to extend the bootstrap confidence interval to more complicated situations.

Define the cumulative distribution function of the bootstrap distribution for the one-parameter models,

$$\hat{G}(s) = \int_{-\infty}^s f_{\hat{\theta}}(\hat{\theta}^*) d\hat{\theta}^* = Pr_{\hat{\theta}}\{\hat{\theta}^* < s\}. \quad (2.16)$$

The integral is replaced by a summation in discrete families.

Suppose that there exists a monotone-increasing transformation g and constants b_0 and a such that

$$\hat{\phi} = g(\hat{\theta}), \quad \phi = g(\theta) \quad (2.17)$$

satisfying

$$\hat{\phi} = \phi + \sigma_{\phi}(Z - b_0), \quad Z \sim N(0, 1) \quad (2.18)$$

with

$$\sigma_{\phi} = 1 + a\phi. \quad (2.19)$$

Assume that $\phi > -1/a$ if $a > 0$ in (2.19), so $\sigma_{\phi} > 0$, and likewise $\phi < -1/a$ if $a < 0$.

Under conditions (2.17)-(2.19), Efron(1987) showed that the BC_a confidence interval of level $1 - 2\alpha$ for θ is given by

$$[\hat{G}^{-1}(\Phi(z[\alpha])), \hat{G}^{-1}(\Phi(z[1 - \alpha]))] \quad (2.20)$$

where

$$z[\alpha] = b_0 + \frac{(b_0 + z_{\alpha})}{1 - a(b_0 + z_{\alpha})}. \quad (2.21)$$

To find BC_a interval we need to calculate the bootstrap distribution $\hat{G}$ and also the two constants b_0 and a . These calculations assume that a transformation g to form (2.18), (2.19) exists, but do not require g to be known. The bias-correction constant b_0 is $b_0 = \Phi^{-1}(\hat{G}(\hat{\theta}))$ under assumptions (2.17)-(2.19). Efron shows that a can be computed by

$$a = \frac{1}{6} SKEW_{\theta=\hat{\theta}}(\dot{l}_{\theta}),$$

where $SKEW_{\theta=\hat{\theta}}(X)$ indicates the skewness of a random variable X evaluated at parameter point θ equal to $\hat{\theta}$, and $\dot{l}_{\theta}$ is the score function of the family $f_{\theta}(\hat{\theta})$,

$$\dot{l}_{\theta}(\hat{\theta}) = \frac{\partial}{\partial \theta} \log f_{\theta}(\hat{\theta}).$$

For nonparametric bootstrap, the acceleration constant a can be approximated by

$$a \doteq \frac{1}{6} \left[\left(\sum_{i=1}^n U_i^3 \right) / \left(\sum_{i=1}^n U_i^2 \right)^{3/2} \right],$$

where

$$U_i = \lim_{\epsilon \rightarrow 0} \frac{t((1-\epsilon)\hat{F}_n + \epsilon\delta_i) - t(\hat{F}_n)}{\epsilon}, \quad i = 1, 2, \dots, n. \quad (2.22)$$

The function U_i is called the empirical influence function of $\hat{\theta} = t(\hat{F}_n)$, where $\hat{F}_n$ is the empirical probability distribution of the sample. The δ_i is a point mass at x_i , so U_i is the derivative of the estimate $\hat{\theta}$ with respect to the mass at point x_i . The U_i 's are computed by substituting a small number like $1/(n+1)$ for ϵ , except in the rare cases they can be calculated exactly.

If both b_0 and a equal 0, then $z[\alpha] = z_{\alpha}$ and (2.20) becomes $[\hat{G}^{-1}(\alpha), \hat{G}^{-1}(1-\alpha)]$. This is the percentile method, described in (2.7). If $a=0$ in (2.21), then (2.20) becomes the BC interval (2.15).

Another intuitive method is the two stage bootstrapping sampling. Since this type of methods is extremely time consuming in simulation study, we choose the prepivoting method (or B-method) by Beran(1987). To establish the confidence interval for a given parameter, Beran(1987) defines a root function $R_n(\theta) = R_n(\mathbf{x}, \theta)$ which should have similar property as a pivotal. Let an estimator of θ be defined as $\hat{\theta} = T(\hat{F}_n)$, where $\hat{F}_n$ is again the empirical distribution from sample of $\mathbf{x}$.

(a) Let $\mathbf{x}_1^*, \dots, \mathbf{x}_M^*$ be M bootstrap samples of size n , each drawn from $\hat{F}_n$. The samples are conditionally independent, given the original sample $\mathbf{x}$. The empirical cdf of the values $\{R_n(\mathbf{x}_j^*, \hat{\theta}_n) : 1 \leq j \leq M\}$ is defined as $\hat{W}_n$.

(b) Let $\hat{F}_{n,j}^*$ denote the estimate of F recomputed from the j th bootstrap sample $\mathbf{x}_j^*$. For every j , let $\mathbf{x}_{j,1}^{**}, \dots, \mathbf{x}_{j,N}^{**}$ be N further bootstrap samples of size n , each drawn from $\hat{F}_{n,j}^*$. These samples are conditionally independent, given the original sample $\mathbf{x}$ and the first-round bootstrap samples $\{\mathbf{x}_j^* : 1 \leq j \leq M\}$. Set $\hat{\theta}_{n,j}^* = T(\hat{F}_{n,j}^*)$. Let Z_j be the fraction of the values $\{R_n(\mathbf{x}_{j,k}^{**}, \hat{\theta}_{n,j}^*) : 1 \leq k \leq N\}$ which are less than $R_n(\mathbf{x}_j^*, \hat{\theta}_n)$. Let the empirical cdf of the $\{Z_j : 1 \leq j \leq M\}$ be $\hat{W}_{n,1}$.

For the mean, there are two choices of $R_n(\mathbf{x}, \theta)$ by Beran(1987),

$$R_n(\mathbf{x}, \mu) \equiv S_n(\mu) = n^{\frac{1}{2}}|\bar{x} - \mu|$$

and

$$R_n(\mathbf{x}, \mu) \equiv T_n(\mu) = n^{\frac{1}{2}}|\bar{x} - \mu|/s.$$

Then the confidence intervals for μ are, respectively,

$$[\bar{x} - \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}/\sqrt{n}, \quad \bar{x} + \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}/\sqrt{n}]$$

and

$$[\bar{x} - \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}s/\sqrt{n}, \quad \bar{x} + \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}s/\sqrt{n}].$$

The function

$$R_n(\mathbf{x}, \sigma^2) \equiv U_n(\sigma^2) = n^{\frac{1}{2}} |s^2 - \sigma^2| \quad (2.23)$$

is used as the root function for σ^2 .

The $100(1 - 2\alpha)\%$ confidence interval for σ^2 is

$$[s^2 - \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}/\sqrt{n}, \quad s^2 + \hat{W}_n^{-1}\{\hat{W}_{n,1}^{-1}(1 - \alpha)\}/\sqrt{n}].$$

The six bootstrap confidence intervals were examined in this dissertation:

- 1) the centralized percentile method,
- 2) the percentile method,
- 3) the BC method,
- 4) the BC_a method,
- 5) the percentile-t method, and
- 6) Beran's B-method. (The bootstrap scale sizes are set at $M = N = 300$.)

There are other bootstrap confidence intervals, but they were not used either because of their nonintuitiveness or because of their constructional difficulties. Moreover, our main goal is to modify those intuitive methods, such as the percentile method.

CHAPTER 3 MODIFIED CONFIDENCE INTERVALS

3.1 Modified Bootstrap Confidence Interval

All of the asymptotic expansions calculated in this section will be in the context of Hall's (1988,1992) smooth function model. To describe the model briefly, assume that the data comprising the sample $\mathcal{X}$ are in the form of n independent and identically distributed d -vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$. Let $\mathbf{X}$ have the distribution of the $\mathbf{X}_i$'s and put $\boldsymbol{\mu} \equiv E(\mathbf{X})$ and $\bar{\mathbf{X}} \equiv n^{-1} \sum_i \mathbf{X}_i$. We suppose that for known real-valued smooth functions f and g , $\theta = f(\boldsymbol{\mu})$ and $\sigma_d^2 = g(\boldsymbol{\mu})$. The estimates of θ and σ_d^2 are $\hat{\theta} \equiv f(\bar{\mathbf{X}})$ and $\hat{\sigma}_d^2 \equiv g(\bar{\mathbf{X}})$, respectively.

By way of regularity conditions, given a d -vector $\mathbf{t} = (t^{(1)}, \dots, t^{(d)})'$, define $\|\mathbf{t}\| = \{(t^{(1)})^2 + \dots + (t^{(d)})^2\}^{1/2}$ and assume Cramér condition, i.e., $\limsup_{\|\mathbf{t}\| \rightarrow \infty} |E\{\exp(i \sum_{j=1}^d t^{(j)} X^{(j)})\}| < 1$, where $t^{(j)}$ and $X^{(j)}$ are the j th elements of d -vectors of $\mathbf{t}$ and $\mathbf{X}$, respectively. This condition holds if the distribution $\mathbf{X}$ is nonsingular. We will assume throughout that $\mathbf{X}$ has sufficiently many finite moments such that the remainder terms in the expansions of coverage probabilities are of the stated order.

Let $x_1, \dots, x_n$ be independent and identically distributed observations from a continuous, univariate population with mean μ and variance σ^2 . Let x have the distribution of the x_i 's. Suppose that we are interested in constructing confidence

intervals for a univariate parameter θ and that the coverage probability of a two-sided $(1 - 2\alpha)$ level confidence interval for θ admits expansion

$$C_1 = (1 - 2\alpha) + n^{-1}2t(z_{1-\alpha})\phi(z_{1-\alpha}) + O(n^{-2}), \quad (3.1)$$

where $z_{1-\alpha}$ is the solution of $\Phi(z_{1-\alpha}) = 1 - \alpha$ and $t(z_{1-\alpha})$ is an odd polynomial with coefficients depending on the moments of x and on the confidence interval method (Hall 1988, 1992). Consider the case of $\theta = \mu$. We have

$$\begin{aligned} t_{Perc-t}(z_{1-\alpha}) &= -\frac{1}{6}(2z_{1-\alpha}^3 + z_{1-\alpha})\kappa_4 - \frac{1}{4}(2z_{1-\alpha}^3 + z_{1-\alpha})\kappa_3^2, \\ t_{Perc}(z_{1-\alpha}) &= -\frac{1}{24}(z_{1-\alpha}^3 + 5z_{1-\alpha})\kappa_4 - \frac{1}{72}(z_{1-\alpha}^5 + 2z_{1-\alpha}^3 - 9z_{1-\alpha})\kappa_3^2 \\ &\quad - \frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3), \\ t_{BC}(z_{1-\alpha}) &= -\frac{1}{24}(z_{1-\alpha}^3 + 13z_{1-\alpha})\kappa_4 - \frac{1}{72}(z_{1-\alpha}^5 - 2z_{1-\alpha}^3 - 41z_{1-\alpha})\kappa_3^2 \\ &\quad - \frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3), \end{aligned}$$

and

$$\begin{aligned} t_{BC_a}(z_{1-\alpha}) &= -\frac{1}{24}(5z_{1-\alpha}^3 + 13z_{1-\alpha})\kappa_4 + \frac{1}{8}(2z_{1-\alpha}^3 + 5z_{1-\alpha})\kappa_3^2 \\ &\quad - \frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3), \end{aligned} \quad (3.2)$$

where κ_3 and κ_4 are third and fourth cumulants of $(x - \mu)/\sigma$, respectively. The subscripts *Perc-t*, *Perc*, *BC*, and *BC_a* in (3.2) refer to the percentile-t method, the percentile method, the BC method, and the *BC_a* method, respectively. In this case, the moment condition on x for being valid expansion (3.1) is $E(|x|^4) < \infty$.

The bootstrap estimate of (3.1) satisfies

$$\hat{C}_1 = (1 - 2\alpha) + n^{-1}2\hat{t}(z_{1-\alpha})\phi(z_{1-\alpha}) + O_p(n^{-2}), \quad (3.3)$$

where $\hat{t}(z_{1-\alpha})$ is obtained from $t(z_{1-\alpha})$ in (3.2) by replacing the population moments with their sample moments (Hall, 1988). For example, if $\theta = \mu$,

$$\hat{t}_{Perc-t}(z_{1-\alpha}) = -\frac{1}{6}(2z_{1-\alpha}^3 + z_{1-\alpha})\hat{\kappa}_4 - \frac{1}{4}(2z_{1-\alpha}^3 + z_{1-\alpha})\hat{\kappa}_3^2,$$

$$\begin{aligned}\hat{t}_{Perc}(z_{1-\alpha}) = & -\frac{1}{24}(z_{1-\alpha}^3 + 5z_{1-\alpha})\hat{\kappa}_4 - \frac{1}{72}(z_{1-\alpha}^5 + 2z_{1-\alpha}^3 - 9z_{1-\alpha})\hat{\kappa}_3^2 \\ & -\frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3),\end{aligned}$$

$$\begin{aligned}\hat{t}_{BC}(z_{1-\alpha}) = & -\frac{1}{24}(z_{1-\alpha}^3 + 13z_{1-\alpha})\hat{\kappa}_4 - \frac{1}{72}(z_{1-\alpha}^5 - 2z_{1-\alpha}^3 - 41z_{1-\alpha})\hat{\kappa}_3^2 \\ & -\frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3),\end{aligned}$$

and

$$\begin{aligned}\hat{t}_{BCa}(z_{1-\alpha}) = & -\frac{1}{24}(5z_{1-\alpha}^3 + 13z_{1-\alpha})\hat{\kappa}_4 + \frac{1}{8}(2z_{1-\alpha}^3 + 5z_{1-\alpha})\hat{\kappa}_3^2 \\ & -\frac{1}{4}z_{1-\alpha}(z_{1-\alpha}^2 + 3),\end{aligned}\tag{3.4}$$

where $\hat{\kappa}_3 = \hat{\sigma}^{-3}n^{-1} \sum_{i=1}^n (x_i - \bar{x})^3$, $\hat{\kappa}_4 = \hat{\sigma}^{-4}n^{-1} \sum_{i=1}^n (x_i - \bar{x})^4 - 3$, $\bar{x} = n^{-1} \sum_{i=1}^n x_i$, and $\hat{\sigma}^2 = n^{-1} \sum_{i=1}^n (x_i - \bar{x})^2$.

In view of (3.1) and (3.2), we may find a distribution in a family of distributions such that the coverage probability C_1 is the closest to $(1 - 2\alpha)$. The distribution is referred to as a preferred distribution. Moreover, we assume that $\mathcal{F}$ is small enough that only one of its members can reach the closest point. In other words, the preferred distribution is assumed to be unique. More precisely, a distribution function F_0 is the preferred distribution in a family of distribution $\mathcal{F}$ if the coverage probability of a two-sided $(1 - 2\alpha)$ level confidence interval for θ has the value of

$$\min_j |(1 - 2\alpha) - C_{1j}| \quad \forall F_j \in \mathcal{F},\tag{3.5}$$

where C_{1j} is the coverage probability (3.1) for the distribution j and $\mathcal{F}$ will be defined later.

Since we feel that the bootstrap procedure “assumes” the data came from the preferred distribution, we also suspect that the further the distribution is from the preferred distribution, the worse is the confidence interval performance. We chose to measure this distance by

$$d(x) = \max_x |F(x) - F_0(x)|, \quad (3.6)$$

where $F_0(x)$ is a preferred distribution function.

Suppose now that a density function $f(x)$ can be expanded formally in a series of derivatives of the standard normal density $\phi(x)$ (Kendall and Stuart, 1979), i.e.,

$$f(x) = \sum_{i=0}^{\infty} c_i H_i(x) \phi(x),$$

where c_i is a constant depending on the moments of x and $H_i(x)$ is Hermite polynomial. Cramér(1963) showed that this expansion is valid if $f(x)$ is a function with a continuous derivative such that $\int_{-\infty}^{\infty} \left(\frac{df}{dx}\right)^2 e^{x^2/2} dx$ converges and if $f(x)$ tends to zero as $|x|$ tends to infinity.

The first six polynomials are

$$H_0 = 1$$

$$H_1 = x$$

$$H_2 = x^2 - 1$$

$$H_3 = x^3 - 3x$$

$$H_4 = x^4 - 6x^2 + 3$$

$$H_5 = x^5 - 10x^3 + 15x$$

$$H_6 = x^6 - 15x^4 + 45x^2 - 15.$$

Then if $f(x)$ is standardized, it may be shown after some algebra that the series becomes

$$f(x) = \phi(x) \left\{ 1 + \frac{\kappa_3}{6} H_3 + \frac{\kappa_4}{24} H_4 + \frac{\kappa_5}{120} H_5 + \frac{\kappa_6 + 10\kappa_3^2}{720} H_6 + \cdots \right\}. \quad (3.7)$$

This is often called the Edgeworth form of Type A series. The distribution function of such a series is easy to obtain. Since

$$\left(-\frac{d}{dx}\right)^r \phi(x) = H_r(x) \phi(x),$$

we have

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \sum_{r=0}^{\infty} c_r \int_{-\infty}^x H_r(x) \phi(x) dx \\ &= \Phi(x) - \sum_{r=1}^{\infty} c_r H_{r-1}(x) \phi(x). \end{aligned}$$

Consequently,

$$\begin{aligned} d(x) &= \max_x |F(x) - F_0(x)| \\ &= \max_x \left| \phi(x) \left\{ \frac{(\kappa'_3 - \kappa_3)}{6} H_2(x) + \frac{(\kappa'_4 - \kappa_4)}{24} H_3(x) + \frac{(\kappa'_5 - \kappa_5)}{120} H_4(x) \right. \right. \\ &\quad \left. \left. + \frac{(\kappa'_6 + 10\kappa_3'^2) - (\kappa_6 + 10\kappa_3^2)}{720} H_5(x) + \cdots \right\} \right|, \end{aligned} \quad (3.8)$$

where κ_i and κ'_i , $i = 3, 4, \dots$, are cumulants for F and F_0 , respectively.

Now, suppose $\mathcal{F}$ is the family of distributions which are included in the Edgeworth form of Type A series with all terms in (3.8) negligible relative to the second term

and $\kappa_4 \geq 0$. Then, in view of (3.2), the preferred distribution for $\theta = \mu$ will be the standard normal distribution.

The distance (3.8) for the distribution j will be

$$d_j(x) = 0.0229\kappa_{4j} + \varepsilon, \quad (3.9)$$

where κ_{4j} is the fourth cumulant of the distribution j and ε will be defined in the following proof.

Proof of the equation (3.9).

From (3.8) under $\mathcal{F}$,

$$\begin{aligned} d_j(x) &= \max_x \left| \phi(x) \left\{ \frac{-\kappa_{4j}}{24} H_3(x) \right\} \right| + \varepsilon \\ &= \frac{\kappa_{4j}}{24} \max_x \left| \phi(x)(x^3 - 3x) \right| + \varepsilon, \end{aligned}$$

where ε is a negligible factor due to the assumptions on $\mathcal{F}$. Let

$$g(x) = \phi(x)(x^3 - 3x).$$

Then,

$$g'(x) = -\phi(x)(x^4 - 6x^2 + 3).$$

Letting $g'(x) = 0$, we have $x = \pm 0.7420$ or ± 2.3344 . Then $\max_x |g(x)| = |g(0.7420)|$ and

$$d_j(x) = \kappa_{4j}|g(0.7420)|/24 + \varepsilon = 0.0229\kappa_{4j} + \varepsilon. \quad \square$$

Consider the case of $\theta = \mu$. For any distribution j , the equations (3.2) can be written as

$$\begin{aligned} t_{Perc-t}(z) &= -a_{1,z}\kappa_{4j} - b_{1,z}\kappa_{3j}^2, \\ t_{Perc}(z) &= -a_{2,z}\kappa_{4j} - b_{2,z}\kappa_{3j}^2 - c_z, \\ t_{BC}(z) &= -a_{3,z}\kappa_{4j} - b_{3,z}\kappa_{3j}^2 - c_z, \end{aligned}$$

and

$$t_{BC_a}(z) = -a_{4,z}\kappa_{4j} + b_{4,z}\kappa_{3j}^2 - c_z, \quad (3.10)$$

where $z \equiv z_{1-\alpha}$, and $\{a_{i,z}, b_{i,z}\}, i = 1, \dots, 4$ and c_z are constants. Note that for any value of z ,

$$a_{i,z} > 0 \quad \forall i \quad \text{and} \quad c_z > 0.$$

Since κ_{3j} is negligible, d and t are correlated, i.e., the further a distribution is from the preferred distribution, the standard normal distribution, the undercoverage of the confidence interval becomes worse.

From (3.10), the general form of $t(z)$ for a distribution j can be written as

$$t_j(z) = -a_z\kappa_{4j} - b_z\kappa_{3j}^2 - c_z, \quad (3.11)$$

$$= -a_z\kappa_{4j} - \delta_1 - c_z, \quad (3.12)$$

where $a_z > 0, c_z \geq 0$, and $\delta_1 \equiv b_z\kappa_{3j}^2$. Since the preferred distribution is the standard normal distribution, we have $\kappa'_3 = 0$. Then, by the definition of $\mathcal{F}$, κ_{3j} is negligible and so δ_1 is negligible for any distribution in $\mathcal{F}$. The subscript j in $t_j(z)$, κ_{3j} , and κ_{4j} always denotes the distribution j and the subscripts $Perc$, $Perc - t$, BC , and BC_a in (3.10) are omitted in the notation $t_j(z)$ since the form of $t(z)$ of each confidence interval in (3.10) do not affect the results derived in this section. Also, when it is not necessary to refer a distribution j , the notation $t(z)$ without any subscript is used.

Let Y_j be the coverage discrepancy of a confidence interval, i.e., $Y_j = 1 - 2\alpha - C_{1j}$. Then, by (3.1) and (3.12),

$$\begin{aligned} Y_j &= 1 - 2\alpha - [1 - 2\alpha + n^{-1}2t_j(z)\phi(z) + O(n^{-2})] \\ &= -n^{-1}2t_j(z)\phi(z) + O(n^{-2}) \end{aligned} \quad (3.13)$$

$$\begin{aligned} &= -n^{-1}2[-a_z\kappa_{4j} - c_z - \delta_1]\phi(z) + O(n^{-2}) \\ &= [n^{-1}2c_z\phi(z) + \delta + O(n^{-2})] + [n^{-1}2a_z\phi(z)]\kappa_{4j}, \end{aligned} \quad (3.14)$$

where $\delta = n^{-1}2\phi(z)\delta_1$ and we have $|\delta| < |\delta_1|$. From (3.14), we see that Y_j is a linear function of κ_{4j} .

Using (3.14) and (3.9), the relationship between Y_j and d_j can be found as

$$\begin{aligned} Y_j &= [n^{-1}2c_z\phi(z) + \delta + \delta_2 + O(n^{-2})] + [n^{-1}2a_z\phi(z)/0.0229]d_j \\ &= \beta_0 + \beta_1d_j, \end{aligned} \quad (3.15)$$

where $\delta_2 = -n^{-1}2a_z\phi(z)\varepsilon/0.0229$, $\beta_0 = n^{-1}2c_z\phi(z) + \delta + \delta_2 + O(n^{-2})$, and $\beta_1 = n^{-1}2a_z\phi(z)/0.0229$. Thus, Y_j is a linear function of d_j with constant coefficients. From (3.13) and (3.15), we have

$$-n^{-1}2t_j(z)\phi(z) + O(n^{-2}) = \beta_0 + \beta_1d_j. \quad (3.16)$$

Since we wish C_1 to be close to $(1 - 2\alpha)$, we may assign a new nominal confidence level $(1 - 2\hat{\alpha})$ so that C_1 is expected to be $(1 - 2\alpha)$. Thus, we let

$$(1 - 2\hat{\alpha}) = (1 - 2\alpha) + \beta_0 + \beta_1d. \quad (3.17)$$

We name the confidence interval constructed by $(1 - 2\hat{\alpha})$ the modified confidence interval. In contrast, the usual $(1 - 2\alpha)$ level confidence interval is called the ordinary confidence interval. Using (3.1), the modified confidence interval has the coverage expansion

$$C_2 = (1 - 2\alpha) + (\beta_0 + \beta_1d) + n^{-1}2t(z')\phi(z') + O(n^{-2}), \quad (3.18)$$

where $z' = z_{1-\alpha+\frac{\beta_0+\beta_1 d_j}{2}}$.

Theorem 1 Assuming the expansion (3.1) and the family of distributions $\mathcal{F}$ defined previously, the modified confidence interval for $\theta = \mu$ has the coverage probability

$$C_2 = (1 - 2\alpha) + O(n^{-2}).$$

The proof of Theorem 1 is given at the end of this section.

Suppose we have k sample observations of Y_j from the family of distributions $\mathcal{F}$. By (3.3), these are the sample estimates of Y_j , i.e., $y_j = -n^{-1}2\hat{t}_j(z)\phi(z) + O_p(n^{-2})$. Suppose we build a simple linear regression model using (3.15),

$$y_j = \beta_0 + \beta_1 d_j + \epsilon_j, \quad j = 1, \dots, k, \quad (3.19)$$

where β_0 and β_1 are unknown parameters and $\epsilon_1, \dots, \epsilon_k$ are the discrepancies between y_j and $\beta_0 + \beta_1 d_j$. Let $\mathcal{F}$ be large and $\{\epsilon_\lambda\}$ denote all the discrepancies for $\lambda \in \mathcal{F}$. We may assume $\epsilon_1, \dots, \epsilon_k$ is a random sample from $\{\epsilon_\lambda\}$. Using the usual sampling theory, we let $E(\epsilon_j) = 0$ and $V(\epsilon_j) = \sigma_\epsilon^2$ and all the ϵ 's are independent. Then, we let

$$(1 - 2\hat{\alpha}) = (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d, \quad (3.20)$$

where $\hat{\beta}_0$ and $\hat{\beta}_1$ are the ordinary least squares estimates of β_0 and β_1 . The bootstrap estimate of C_2 satisfies

$$\hat{C}_2 = (1 - 2\alpha) + (\hat{\beta}_0 + \hat{\beta}_1 d) + n^{-1}2\hat{t}(\hat{z}')\phi(\hat{z}') + O_p(n^{-2}), \quad (3.21)$$

where $\hat{z}' = z_{1-\alpha+\frac{\hat{\beta}_0+\hat{\beta}_1 d_j}{2}}$ and $\hat{t}(\hat{z}')$ is obtained from $t(\hat{z}')$ by replacing population moments with their sample moments. The notation $\hat{C}_{2j}$ will be used for $\hat{C}_2$ for the distribution j .

Note that from (3.3), the bootstrap estimate of the ordinary confidence interval has the mean coverage probability

$$E[\hat{C}_1] = (1 - 2\alpha) + O(n^{-1}). \quad (3.22)$$

The proof of (3.22) is given at the end of this section.

Theorem 2 Assume the expansion (3.1), the regression model (3.19), and the family of distributions $\mathcal{F}$ defined previously. Assume further that $\max_{1 \leq i \leq k} c_{ki}^2 \rightarrow 0$ as $k \rightarrow \infty$, $\lim_{k \rightarrow \infty} \bar{d}_k = \bar{d} < \infty$, and $\lim_{k \rightarrow \infty} k^{-1} w_k = w < \infty$, where $c_{ki} = (d_i - \bar{d}_k) / \sqrt{w_k}$ and $w_k = \sum_{i=1}^k (d_i - \bar{d}_k)^2$. Then, the bootstrap estimate of the modified confidence interval for $\theta = \mu$ has the mean coverage probability

$$E[\hat{C}_2] = (1 - 2\alpha) + O(n^{-2}) \quad \text{as } k \rightarrow \infty.$$

Note that from (3.22) the bootstrap estimate the ordinary confidence interval has the mean coverage error $O(n^{-1})$. Therefore, the modified confidence interval has reduced the mean coverage error from $O(n^{-1})$ to $O(n^{-2})$.

However, if the cumulants above the fourth are neglected, two disadvantages of the series (3.7) in representing probability distributions are: (a) that $f(x)$ may be negative for some values of x and (b) that the curve may be multimodal (Draper and Tierney, 1972; Barton and Dennis, 1952).

Now, consider another family of distribution $\mathcal{F}_r$. Define $\mathcal{F}_r$ as the family of the positive and unimodal distributions which are included in the Edgeworth form of Type A series with all terms in (3.8) negligible relative to the terms containing κ_3 and κ_4 . Then, in view of (3.2), the preferred distribution for $\theta = \mu$ will be the standard normal distribution.

In this case, from (3.8) we have

$$d(x) = \max_x \left| \phi(x) \left\{ \kappa_3 H_2(x)/6 + \kappa_4 H_3(x)/24 + \kappa_3^2 H_5(x)/72 \right\} \right|. \quad (3.23)$$

Draper and Tierney showed that the positive and unimodal Edgeworth region is that enclosed by the curve connecting the highest points at the each value of κ_4 and the κ_4 axis in Figure 3.1. To see the relationship between d and $-t(z)$, 265 pairs of

(κ_3, κ_4) were selected, which lies within the positive and unimodal Edgeworth region. Specifically, for a fixed value of κ_4 , $\kappa_4 = 0.0, 0.1, 0.2, \dots, 2.3$, 11 values of κ_3 were selected such that these values are equally spaced, and finally the pair $(\kappa_3, \kappa_4) = (0, 2.4)$ was also selected. The selected combinations can be found in Figure 3.1. Figure 3.2 shows the results for the 90% confidence intervals using (3.2) and (3.23). There are almost linear relationships, i.e., the further a distribution is from the preferred distribution, the undercoverage of the confidence interval becomes worse. The estimated correlation coefficients based on the 265 observations (or distributions) are 0.989, 0.971, 0.945, and 0.934 for the percentile-t, the percentile, the BC, and the BC_a method, respectively. Note that the highest points at the each value of d are for $\kappa_3 = 0$, i.e., these points are for the family of distributions $\mathcal{F}$ defined previously. We can see the linear relationship between d and $-t(z)$ for $\mathcal{F}$. Figure 3.3 shows the similar results for the 95% confidence intervals. The estimated correlation coefficients are 0.989, 0.977, 0.950, and 0.933 for the percentile-t, the percentile, the BC, and the BC_a method, respectively. Assuming the expansion (3.1) and the regression model (3.19), Theorem 2 holds under this $\mathcal{F}_7$.

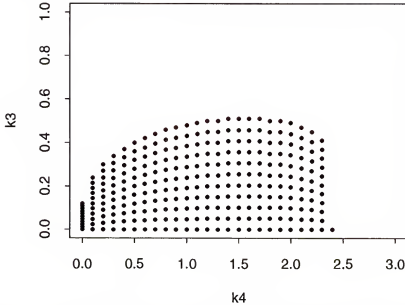


Figure 3.1. Positive and unimodal Edgeworth region & 265 distributions.

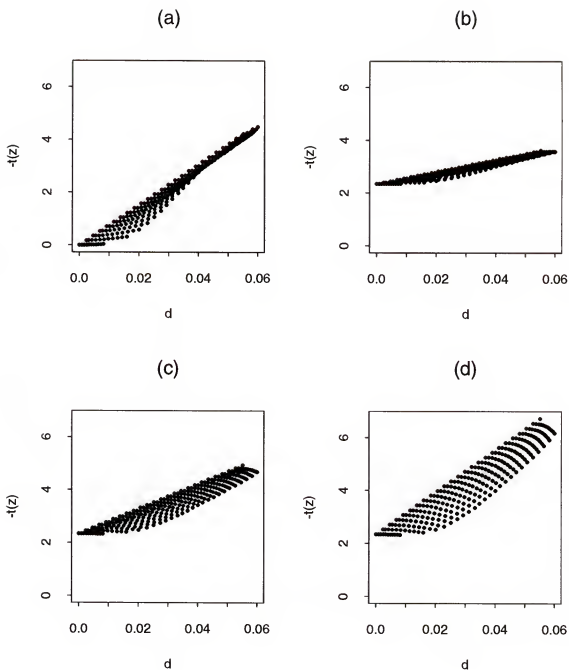


Figure 3.2. Relationship between d and $t(z)$ for a 90% confidence interval for the mean. $n = 265$ (distributions). (a) percentile-t method, $\hat{\rho} = 0.989$, (b) percentile method, $\hat{\rho} = 0.971$, (c) BC method, $\hat{\rho} = 0.945$, and (d) BC_a method, $\hat{\rho} = 0.934$.

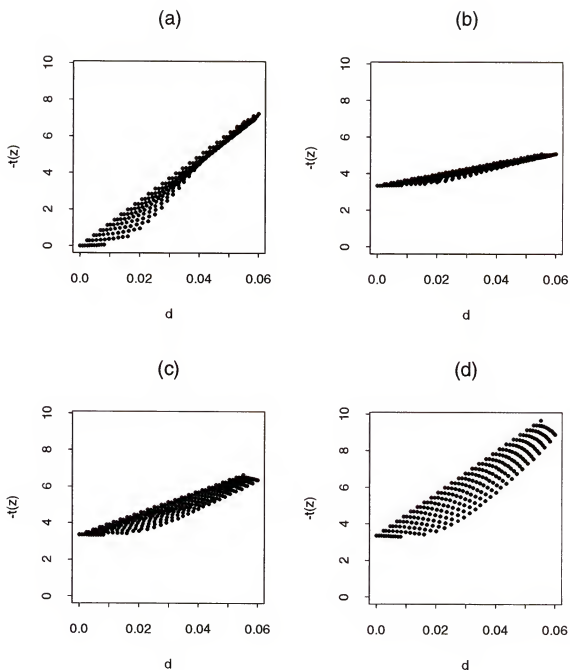


Figure 3.3. Relationship between d and $t(z)$ for a 95% confidence interval for the mean. $n = 265$ (distributions). (a) percentile-t method, $\hat{\rho} = 0.989$, (b) percentile method, $\hat{\rho} = 0.977$, (c) BC method, $\hat{\rho} = 0.950$, and (d) BC_a method, $\hat{\rho} = 0.933$.

In practice, we chose to measure the distance (3.6) by the Kolmogorov-Smirnov distance(KSD). Since the distributions have different location and scale parameters, the standardized empirical distribution is used for the KSD. More precisely, we let the preferred distribution $F_0(x)$ be standardized so that it has 0 mean and unit variance. Let $\bar{x}$ and s^2 be, respectively, the sample mean and variance of $\mathbf{x}$. Define the standardized data as $\mathbf{x}^0 = (\mathbf{x} - \bar{x})/s$ and $\hat{F}_n^0(x)$ as the empirical distribution based on $\mathbf{x}^0$. Then the expected standardized KSD (ESKSD) between the preferred distribution $F_0(x)$ and $F(x)$ with a sample of size n is defined as

$$ESKSD_n = E_F[\sup_x |\hat{F}_n^0(x) - F_0(x)|]. \quad (3.24)$$

Proof of Theorem 1.

Let

$$z_x = \Phi^{-1}(x) \quad \text{and} \quad \phi(z_x) = \frac{1}{\sqrt{2\pi}} \exp(-z_x^2/2).$$

From (3.16), we have

$$\beta_0 + \beta_1 d = O(n^{-1}). \quad (3.25)$$

Then, by Taylor expansion and (3.25),

$$\begin{aligned} \Phi^{-1}\left(x + \frac{\beta_0 + \beta_1 d}{2}\right) &= \Phi^{-1}(x) + \frac{d}{dx} \left[\Phi^{-1}(x) \right] \left(x + \frac{\beta_0 + \beta_1 d}{2} - x \right) + O(n^{-2}) \\ &= z_x + O(n^{-1}). \end{aligned}$$

Taking $x = 1 - \alpha$, we have

$$z' = z + O(n^{-1}), \quad (3.26)$$

where $z' = z_{1-\alpha + \frac{\beta_0 + \beta_1 d}{2}}$.

By Taylor expansion,

$$\begin{aligned}
\phi\left(z_{x+\frac{\beta_0+\beta_1d}{2}}\right) &= \phi(z_x) + \frac{d}{dz_x}\left[\phi(z_x)\right]\left(z_{x+\frac{\beta_0+\beta_1d}{2}} - z_x\right) + O(n^{-2}) \\
&= \phi(z_x) + \frac{d}{dz_x}\left[\phi(z_x)\right]\left(z_x + O(n^{-1}) - z_x\right) + O(n^{-2}) \\
&= \phi(z_x) + O(n^{-1}).
\end{aligned}$$

Taking $x = 1 - \alpha$, we have

$$\phi(z') = \phi(z) + O(n^{-1}). \quad (3.27)$$

Also, using (3.26) and (3.2), we have

$$t(z') = t(z) + O(n^{-1}). \quad (3.28)$$

Using (3.27), (3.28), and (3.16), the modified coverage expansion (3.18) can be written as

$$\begin{aligned}
C_2 &= (1 - 2\alpha) + \beta_0 + \beta_1 d + [n^{-1}2t(z')\phi(z')] + O(n^{-2}) \\
&= (1 - 2\alpha) + \beta_0 + \beta_1 d + [n^{-1}2t(z)\phi(z)] + O(n^{-2}) \\
&= (1 - 2\alpha) - n^{-1}2t(z)\phi(z) + n^{-1}2t(z)\phi(z) + O(n^{-2}) \\
&= (1 - 2\alpha) + O(n^{-2}). \square
\end{aligned}$$

Proof of the equation (3.22).

We can expand (3.3) as follows:

$$\begin{aligned}
\hat{C}_1 &= (1 - 2\alpha) + n^{-1}2\hat{t}(z)\phi(z) + O_p(n^{-2}) \\
&= (1 - 2\alpha) + n^{-1}2\hat{t}(z)\phi(z) + n^{-2}2\hat{s}(z)\phi(z) + O_p(n^{-3}) \\
&= (1 - 2\alpha) + n^{-3/2}2\phi(z)[n^{1/2}\hat{t}(z) + n^{-1/2}\hat{s}(z)] + O_p(n^{-3}) \\
&= (1 - 2\alpha) + n^{-3/2}2\phi(z)\left\{n^{1/2}[\hat{t}(z) - t(z)] + n^{-1}n^{1/2}[\hat{s}(z) - s(z)]\right\}
\end{aligned}$$

$$+n^{-3/2}2\phi(z)\left\{n^{1/2}t(z)+n^{-1/2}s(z)\right\}+O_p(n^{-3}) \quad (3.29)$$

for a polynomial $\hat{s}(z)$ which is also a function of the sample moments.

Let G be the joint distribution function of $x_1, \dots, x_n$. Let

$$\begin{aligned} T_n &= n^{1/2}[\hat{t}(z) - t(z)] + n^{-1}n^{1/2}[\hat{s}(z) - s(z)] \\ &= S_n + n^{-1}\Delta_n, \end{aligned}$$

where $S_n = n^{1/2}[\hat{t}(z) - t(z)]$ and $\Delta_n = n^{1/2}[\hat{s}(z) - s(z)]$. Note that $\Delta_n = O_p(1)$ and moments and cumulants of T_n differ from those of S_n only in terms of n^{-1} or smaller. Furthermore, odd-order moments and cumulants of T_n admit expansions in odd powers of $n^{-1/2}$ (Hall, 1992). Thus

$$E_G(T_n) = E_G(S_n) + O(n^{-3/2}). \quad (3.30)$$

Also, note that

$$\begin{aligned} E_G[\hat{\kappa}_3] &= \kappa_3 + O(n^{-1}), \\ E_G[\hat{\kappa}_4] &= \kappa_4 + O(n^{-1}), \end{aligned}$$

and

$$E_G[\hat{\kappa}_3^2] = \kappa_3^2 + O(n^{-1}), \quad \text{since } V(\hat{\kappa}_3) = O(n^{-1}). \quad (3.31)$$

Then, using (3.11) and (3.31), we have

$$\begin{aligned} E_G[\hat{t}(z)] &= E_G[-a_z\hat{\kappa}_4 - b_z\hat{\kappa}_3^2 - c_z] \\ &= (-a_z\kappa_4 - b_z\kappa_3^2 - c_z) + O(n^{-1}) \\ &= t(z) + O(n^{-1}). \end{aligned} \quad (3.32)$$

Taking expectations on (3.29) with respect to G , then by (3.30) and (3.32), we have

$$\begin{aligned}
E_G[\hat{C}_1] &= (1 - 2\alpha) + n^{-3/2} 2\phi(z) \left\{ E_G[n^{1/2}(\hat{t}(z) - t(z))] + O(n^{-3/2}) \right\} \\
&\quad + n^{-3/2} 2\phi(z) \left\{ n^{1/2} t(z) + n^{-1/2} s(z) \right\} + E_G[O_p(n^{-3})] \\
&= (1 - 2\alpha) + n^{-3/2} 2\phi(z) \left\{ n^{1/2} t(z) + n^{-1/2} s(z) \right\} + O(n^{-2}) + E_G[O_p(n^{-3})] \\
&= (1 - 2\alpha) + n^{-1} 2\phi(z) t(z) + O(n^{-2}) + E_G[O_p(n^{-3})].
\end{aligned}$$

Since we can deduce $E_G[O_p(n^{-3})] = O(n^{-3})$ by using the same arguments in (3.29) and (3.30), then we have

$$\begin{aligned}
E_G[\hat{C}_1] &= (1 - 2\alpha) + n^{-1} 2\phi(z) t(z) + O(n^{-2}) \\
&= (1 - 2\alpha) + O(n^{-1}). \quad \square
\end{aligned} \tag{3.33}$$

Proof of Theorem 2.

The ordinary least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{d} \quad \text{and} \\
\hat{\beta}_1 &= \frac{\sum_{i=1}^k y_i (d_i - \bar{d})}{\sum_{i=1}^k (d_i - \bar{d})^2}.
\end{aligned}$$

Under the assumptions on d_i , $\sqrt{k}\{(\hat{\beta}_0, \hat{\beta}_1) - (\beta_0, \beta_1)\}$ is asymptotically bivariate normal with mean $\mathbf{0}$ and variance-covariance matrix $\sigma_\epsilon^2 \mathbf{V}^{-1}$, which has elements $v_{11} = 1 + \bar{d}^2/w$, $v_{12} = v_{21} = -\bar{d}/w$, and $v_{22} = 1/w$ (Sen and Singer, 1993). Then we have

$$\begin{aligned}
\hat{\beta}_0 &= \beta_0 + O_p(k^{-1/2}), \\
\hat{\beta}_1 &= \beta_1 + O_p(k^{-1/2}),
\end{aligned}$$

and

$$(3.34) \quad \beta_0 + \beta_1 d_j + \beta_0 + \beta_1 d_j + O^p(k_{-1/2}).$$

Let

$$z_x = \Phi^{-1}(x) \quad \text{and} \quad \phi(z) = \frac{\sqrt{2}}{1} \exp(-z^2/2).$$

Then, by Taylor expansion and (3.34),

$$\begin{aligned} \Phi^{-1}\left(x + \frac{2}{\beta_1 d_j + \beta_0}\right) &= \Phi^{-1}(x) + \frac{p}{\beta_1} \left[\Phi^{-1}(x) \right] \left(x - \frac{2}{\beta_1 d_j + \beta_0} \right) \\ &= \left(x - \frac{2}{\beta_1 d_j + \beta_0} + x \right)^{x=0} \left[\Phi^{-1}(x) \right] \frac{z^{xp} 2}{p^2} + \end{aligned}$$

$$= xz + (\beta_0 + \beta_1 d_j p/2 + \beta_0 + \beta_1 d_j q_c/4 + O^p(k_{-1/2}^d),$$

where $p^x = \left[\Phi^{-1}(x) \right] \frac{z^{xp}}{p}$, $q_c = \frac{2}{p^2} \left[\Phi^{-1}(x) \right] \frac{z^{xp}}{p}$, and c is between x and $x + [\beta_0 + \beta_1 d_j]/2$.

Taking $x = 1 - \alpha$, we have

$$(3.35) \quad z_j' = (\beta_0 + \beta_1 d_j p_{1-\alpha}/2 + \beta_0 + \beta_1 d_j q_c/4 + O^p(k_{-1/2}^d),$$

where $z_j' = z^{-1} + \frac{z}{p^2} \frac{z^{xp} 2}{p}$ and $z^{-1} = z^{-\alpha}$.

Again by Taylor expansion, (3.34), and (3.35),

$$\left(xz - \frac{z}{p^2} \frac{z^{xp} 2}{p} + (xz)\phi \right) = \left(xz - \frac{z}{p^2} \frac{z^{xp} 2}{p} + (xz)\phi \right)$$

$$\left(xz - \frac{z}{p^2} \frac{z^{xp} 2}{p} + (xz)\phi \right)^{p=x} \left[\left(xz \right) \phi \right] \frac{z^{xp} 2}{p^2} +$$

$$\left(4^{1/2} \bar{b}_z (f_j p_1 g' + g_j) + z/2 d(f_j p_1 g' + g_j) + (xz)\phi \right) \left[\left(xz \right) \phi \right] \frac{z^{xp}}{p} +$$

$$\left(xz - \frac{z}{p^2} \frac{z^{xp} 2}{p} + (xz)\phi \right)^{p=x} \left[\left(xz \right) \phi \right] \frac{z^{xp} 2}{p^2} + \left(xz - (z_{1/1} \mathcal{Y})^d O + \right.$$

$$\left. 4^{1/2} \bar{b}_z (f_j p_1 g' + g_j) + z/2 d(f_j p_1 g' + g_j) \right) + (xz)\phi =$$

$$+ (z_{1/1} \mathcal{Y})^d O + p m_z \left[4^{1/2} \bar{b}_z (f_j p_1 g' + g_j) + z/2 d(f_j p_1 g' + g_j) \right] +$$

where $r_x = \frac{d}{dz_x}[\phi(z_x)]$, $w_d = \frac{1}{2} \frac{d^2}{dz_x^2}[\phi(z_x)]|_{x=d}$, and d is between z_x and $z_{(x+(\hat{\beta}_0+\hat{\beta}_1 d_j)/2)}$.

Taking $x = 1 - \alpha$, we have

$$\begin{aligned} \phi(\hat{z}') &= \phi(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]r_{1-\alpha} \\ &\quad + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]^2 w_d + O_p(k^{-1/2}). \end{aligned} \quad (3.36)$$

Also, using (3.35) and (3.4), we have

$$\hat{t}_j(\hat{z}') = \hat{t}_j(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]\hat{t}_{0j}(z) + O_p(k^{-1/2}), \quad (3.37)$$

where $\hat{t}_{0j}(z)$ is a polynomial with coefficients depending on the sample moments.

(The derivation of $\hat{t}_{0j}(z)$ is given at the end of this section.)

Using (3.36) and (3.37), the modified coverage expansion (3.21) can be written as

$$\begin{aligned} \hat{C}_{2j} &= (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j + [n^{-1} 2\hat{t}_j(\hat{z}')\phi(\hat{z}')] + O_p(n^{-2}) \\ &= (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j \\ &\quad + n^{-1} 2 \left\{ \hat{t}_j(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]\hat{t}_{0j}(z) + O_p(k^{-1/2}) \right\} \\ &\quad \times \left\{ \phi(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]r_{1-\alpha} \right. \\ &\quad \left. + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]^2 w_d + O_p(k^{-1/2}) \right\} + O_p(n^{-2}). \end{aligned} \quad (3.38)$$

Let $k > n^2$. Then, we have

$$\begin{aligned} \hat{C}_{2j} &= (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j \\ &\quad + n^{-1} 2 \left\{ \hat{t}_j(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]\hat{t}_{0j}(z) \right\} \\ &\quad \times \left\{ \phi(z) + [(\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]r_{1-\alpha} \right. \end{aligned}$$

$$+ \left[(\beta_0 + \beta_1 d_j) p_{1-\alpha} / 2 + (\beta_0 + \beta_1 d_j)^2 q_c / 4 \right]^2 w_d \Big\} + O_p(n^{-2}). \quad (3.39)$$

Note that from (3.16), we have

$$\beta_0 + \beta_1 d_j = O(n^{-1})$$

and since $\hat{t}_j(z)$ and $\hat{t}_{0j}(z)$ are functions of sample moments, we have

$$\hat{t}_j(z) = t_j(z) + O_p(n^{-1/2})$$

and

$$\hat{t}_{0j}(z) = t_{0j}(z) + O_p(n^{-1/2}).$$

Then, (3.39) is reduced to

$$\hat{C}_{2j} = (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j + n^{-1} 2\hat{t}_j(z) \phi(z) + O_p(n^{-2}). \quad (3.40)$$

Let $(x_{1j}, \dots, x_{nj})$ be iid samples from a distribution j and G_j be the joint distribution function of $x_{1j}, \dots, x_{nj}$ and K be the joint distribution function of $y_1, \dots, y_k$ of (3.19). Let H be the joint distribution function of $x_{1j}, \dots, x_{nj}$ and $y_1, \dots, y_k$. Note that since $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimates of β_0 and β_1 , we have

$$E_K(\hat{\beta}_0 + \hat{\beta}_1 d_j) = \beta_0 + \beta_1 d_j,$$

and from (3.33) for a distribution j , we have

$$E_{G_j}[n^{-1} 2\hat{t}_j(z) \phi(z) + O_p(n^{-2})] = n^{-1} 2t_j(z) \phi(z) + O(n^{-2}).$$

Taking expectations on (3.40) with respect to H , then by (3.16), we have

$$\begin{aligned} E_H(\hat{C}_{2j}) &= E_H[(1 - 2\alpha) + (\hat{\beta}_0 + \hat{\beta}_1 d_j) + n^{-1} 2\hat{t}_j(z) \phi(z) + O_p(n^{-2})] \\ &= (1 - 2\alpha) + E_K[\hat{\beta}_0 + \hat{\beta}_1 d_j] + E_{G_j}[n^{-1} 2\hat{t}_j(z) \phi(z) + O_p(n^{-2})] \\ &= (1 - 2\alpha) + (\beta_0 + \beta_1 d_j) + n^{-1} 2t_j(z) \phi(z) + O(n^{-2}) \\ &= (1 - 2\alpha) - n^{-1} 2t_j(z) \phi(z) + n^{-1} 2t_j(z) \phi(z) + O(n^{-2}) \end{aligned}$$

$$= (1 - 2\alpha) + O(n^{-2}). \quad (3.41)$$

Now, let $k \leq n^2$. From (3.38), we have

$$\begin{aligned} \hat{C}_{2j} &= (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j \\ &\quad + n^{-1} 2 \left\{ \hat{t}_j(z) + [(\beta_0 + \beta_1 d_j) p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4] \hat{t}_{0j}(z) \right\} \\ &\quad \times \left\{ \phi(z) + [(\beta_0 + \beta_1 d_j) p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4] r_{1-\alpha} \right. \\ &\quad \left. + [(\beta_0 + \beta_1 d_j) p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4]^2 w_d \right\} + n^{-1} O_p(k^{-1/2}) + O_p(n^{-2}). \end{aligned} \quad (3.42)$$

By the same arguments before (3.40), (3.42) is reduced to

$$\hat{C}_{2j} = (1 - 2\alpha) + \hat{\beta}_0 + \hat{\beta}_1 d_j + n^{-1} [2\hat{t}_j(z)\phi(z) + O_p(k^{-1/2})] + O_p(n^{-2}). \quad (3.43)$$

Since $O_p(k^{-1/2})$ is calculated from the fact $\hat{\beta}_0 + \hat{\beta}_1 d_j = \beta_0 + \beta_1 d_j + O_p(k^{-1/2})$, we have

$$\begin{aligned} E_K[O_p(k^{-1/2})] &= E_K[\hat{\beta}_0 + \hat{\beta}_1 d_j - (\beta_0 + \beta_1 d_j)] \\ &= \beta_0 + \beta_1 d_j - (\beta_0 + \beta_1 d_j) \\ &= 0. \end{aligned}$$

Taking expectations on (3.43) with respect to H , then by (3.16), we have

$$\begin{aligned} E_H(\hat{C}_{2j}) &= E_H[(1 - 2\alpha) + (\hat{\beta}_0 + \hat{\beta}_1 d_j) + n^{-1} [2\hat{t}_j(z)\phi(z) + O_p(k^{-1/2})] + O_p(n^{-2})] \\ &= (1 - 2\alpha) + E_K[\hat{\beta}_0 + \hat{\beta}_1 d_j] + E_{G_j}[n^{-1} 2\hat{t}_j(z)\phi(z) + O_p(n^{-2})] \\ &= (1 - 2\alpha) + (\beta_0 + \beta_1 d_j) + n^{-1} 2t_j(z)\phi(z) + O(n^{-2}) \\ &= (1 - 2\alpha) - n^{-1} 2t_j(z)\phi(z) + n^{-1} 2t_j(z)\phi(z) + O(n^{-2}) \\ &= (1 - 2\alpha) + O(n^{-2}). \end{aligned} \quad (3.44)$$

The proof is completed by (3.41) and (3.44). $\square$

Derivation $\hat{t}_{0j}(z)$ in the equation (3.37).

Consider the case of the percentile interval, since the calculations for the other intervals are similar. Write $\hat{t}_j(z)$ and $\hat{t}_j(z')$ for the percentile interval using (3.4)

$$\hat{t}_j(z) = -\frac{1}{24}(z^3 + 5z)\hat{\kappa}_{4j} - \frac{1}{72}(z^5 + 2z^3 - 9z)\hat{\kappa}_{3j}^2 - \frac{1}{4}z(z^2 + 3)$$

and

$$\hat{t}_j(z') = -\frac{1}{24}[(z')^3 + 5z']\hat{\kappa}_{4j} - \frac{1}{72}[(z')^5 + 2(z')^3 - 9z']\hat{\kappa}_{3j}^2 - \frac{1}{4}z'[(z')^2 + 3], \quad (3.45)$$

where $z = z_{1-\alpha}$ and $z' = z_{1-\alpha + \frac{\beta_0 + \beta_1 d_j}{2}}$.

Since, from (3.35),

$$\begin{aligned} z' &= z + (\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4 + O_p(k^{-1/2}) \\ &\equiv (z + a) + O_p(k^{-1/2}) \\ &\equiv z'' + O_p(k^{-1/2}), \end{aligned}$$

where $a \equiv (\beta_0 + \beta_1 d_j)p_{1-\alpha}/2 + (\beta_0 + \beta_1 d_j)^2 q_c/4$ and $z'' \equiv z + a$,

by substituting $z' = z'' + O_p(k^{-1/2})$ to the right hand side of (3.45), we have

$$\hat{t}_j(z') = -\frac{1}{24}(z''^3 + 5z'')\hat{\kappa}_{4j} - \frac{1}{72}(z''^5 + 2z''^3 - 9z'')\hat{\kappa}_{3j}^2 - \frac{1}{4}z''(z''^2 + 3) + O_p(k^{-1/2}), \quad (3.46)$$

using the facts, $\hat{\kappa}_{3j} = \kappa_{3j} + O_p(n^{-1/2})$ and $\hat{\kappa}_{4j} = \kappa_{4j} + O_p(n^{-1/2})$.

Substituting $z'' (\equiv z + a)$ to the right hand side of (3.46),

$$\begin{aligned}
\hat{t}_j(z') &= \hat{t}_j(z) - \frac{1}{24} \left[(3z^2a + 3za^2 + a^3) + 5a \right] \hat{\kappa}_{4j} \\
&\quad - \frac{1}{72} \left[(5z^4a + 10z^3a^2 + 10z^2a^3 + 5za^4 + a^5) + 2(3z^2a + 3za^2 + a^3) - 9a \right] \hat{\kappa}_{3j}^2 \\
&\quad - \frac{1}{4} \left[(3z^2a + 3za^2 + a^3) + 3a \right] + O_p(k^{-1/2}), \\
&= \hat{t}_j(z) + a \left\{ -\frac{1}{24} \left[(3z^2 + 3za + a^2) + 5 \right] \hat{\kappa}_{4j} \right. \\
&\quad \left. - \frac{1}{72} \left[(5z^4 + 10z^3a + 10z^2a^2 + 5za^3 + a^4) + 2(3z^2 + 3za + a^2) - 9 \right] \hat{\kappa}_{3j}^2 \right. \\
&\quad \left. - \frac{1}{4} \left[(3z^2 + 3za + a^2) + 3 \right] \right\} + O_p(k^{-1/2}).
\end{aligned}$$

Letting the terms in the big brackets be $\hat{t}_{0j}(z)$, we have

$$\begin{aligned}
\hat{t}_j(z') &= \hat{t}_j(z) + a \left\{ \hat{t}_{0j}(z) \right\} + O_p(k^{-1/2}) \\
&= \hat{t}_j(z) + [(\beta_0 + \beta_1 d_j) p_{1-\alpha} / 2 + (\beta_0 + \beta_1 d_j)^2 q_c / 4] \hat{t}_{0j}(z) + O_p(k^{-1/2}),
\end{aligned}$$

which is the equation (3.37). $\square$

3.2 Practical Consideration

The choice of distributions to estimate the parameters in (3.19) would be a problem in practice because it cannot exhaust all the distributions. If a statistician has no idea at all on where the sample might come from, then the method suggested in the

previous section is of little help and we do not know how to find a good confidence level for a small sample. Actually, the data might come from a Cauchy distribution which makes the confidence interval for the mean meaningless. But if we have some idea of what the true distribution is, then a list of possible distributions should not be too difficult to construct. In this case, the modified bootstrap confidence interval is useful because parametric estimates cannot cover a large variety of parametric families. However, the list still cannot be exhaustive. The following two theorems can alleviate this problem.

Theorem 3. If $X_1, X_2, \dots, X_n$ from $F(x)$ gets a coverage $\hat{\pi}$ for $\theta = \mu$ by using (3.17), then data $Y_1, Y_2, \dots, Y_n$ from $F(\frac{y-\mu}{\sigma})$ for any μ and σ will have the same coverage $\hat{\pi}$.

This theorem shows that we need only to use $N(0, 1)$ for all normal distributions and $U(-1, 1)$ for all uniform distributions.

Proof.

Since the KSD is standardized by the sample mean and the sample standard deviation, it is not affected by the location and scalar transformation (LST). Again, because the order of the data is invariant under LST, the coverage remains the same for $\{X_i\}$ and $\{Y_i = \sigma X_i + \mu\}$. $\square$

Theorem 4. Let F_i and F_j be two distributions such that F_i is used in the list to build the regression equation (3.19), but F_j is not in the list. Suppose when (3.17) is used, the estimated modified coverage for a sample of size n from F_i is $\hat{\pi}_{n,i}$ and from F_j is $\hat{\pi}_{n,j}$. Then for any $\epsilon > 0$, there exists a δ and N such that for $KSD(F_i, F_j) < \delta$ and $n > N$, $|\hat{\pi}_{n,i} - \hat{\pi}_{n,j}| < \epsilon$.

Proof.

Let the new nominal levels of the modified confidence intervals for the distribution

F_i and F_j be, respectively,

$$\pi_i = (1 - 2\alpha) + \beta_0 + \beta_1 d_i,$$

$$\pi_j = (1 - 2\alpha) + \beta_0 + \beta_1 d_j,$$

and $\hat{\pi}_{n,i}$ and $\hat{\pi}_{n,j}$ be the true coverage probabilities of the modified confidence interval.

Then, by (3.3), we have $\hat{\pi}_{n,i} \rightarrow \pi_i$ and $\hat{\pi}_{n,j} \rightarrow \pi_j$.

For fixed ϵ, δ and N can be found as follows:

There exists an N_1 such that

$$n > N_1, \quad |\hat{\pi}_{n,i} - \pi_i| < \epsilon/3 \text{ and } |\hat{\pi}_{n,j} - \pi_j| < \epsilon/3.$$

Let δ be chosen so that $\delta < \epsilon/(9|\beta_1|)$. There exists N_2 such that

$$n > N_2, \quad |d_i - KSD(F_i, F_0)| < \epsilon/(9|\beta_1|) \text{ and } |d_j - KSD(F_j, F_0)| < \epsilon/(9|\beta_1|),$$

where F_0 denotes a preferred distribution. Note

$$|d_i - d_j| \leq |d_i - KSD(F_i, F_0)| + |d_j - KSD(F_i, F_0)| + |KSD(F_i, F_0) - KSD(F_j, F_0)|$$

and by $|KSD(F_i, F_0) - KSD(F_j, F_0)| \leq KSD(F_i, F_j)$, we have $|d_i - d_j| \leq \epsilon/(3|\beta_1|)$,

when $KSD(F_i, F_j) < \delta$ and $N > \max(N_1, N_2)$.

By $|\pi_i - \pi_j| = |\beta_1||d_i - d_j|$, we have $|\pi_i - \pi_j| \leq \epsilon/3$ and

$$|\hat{\pi}_{n,i} - \hat{\pi}_{n,j}| \leq |\hat{\pi}_{n,i} - \pi_i| + |\pi_i - \pi_j| + |\pi_i - \hat{\pi}_{n,j}| \leq \epsilon. \square$$

This theorem guarantees that the list need only to be representative, not exhaustive, because for a sufficiently rich list there will be at least one distribution F_i in the list for which a new distribution F_j is sufficiently close to F_i in KSD.

CHAPTER 4 SMALL SAMPLE SIMULATION STUDY

4.1 Simulation Plan

Our initial simulation study contained the 10 distributions listed below.

1. Normal $N(0, 1)$
2. Uniform $(-1, 1)$
3. Mixed normal I: 25% $N(0, 1)$ and 75% $N(0, 9)$
4. Mixed normal II: 75% $N(0, 1)$ and 25% $N(0, 9)$
5. χ_4^2 (List 1)
6. Double exponential (each side with mean 1)
7. Exponential (1)
8. t_2 for the mean (t_4 for the variance)
9. Lognormal with $N(0,1)$
10. Weibull with density function $bx^{b-1} \exp(-x^b)$ with $b = 1/2$, $0 < x < \infty$

These distributions are used to estimate the parameters of (3.19). The choice of these ten distributions was somewhat arbitrary because there seemed no objective way to choose them. Some of them have been used as alternatives to the normal distribution (e.g., 2 and 5 in Box et al.(1978)), and others have been used to evaluate the robustness of the bootstrap procedures (e.g., 1, 2 and 4 in Peters and Freedman(1987), 9 in Noreen(1989), 1, 7, 8 and 10 by Loh and Wu(1987)). The Cauchy distribution

was not used because it does not have a mean or a variance. The beta distribution is not in this list because it was reserved for later use as an outside evaluator for a new confidence interval method based on this list.

There is a possibility that $(1 - 2\hat{\alpha}) \geq 1$. The first approach is to extrapolate the values outside the data range. The second approach is to restrict $(1 - 2\hat{\alpha}) \leq 1$, i.e., we assign the new confidence level 1 when $(1 - 2\hat{\alpha}) \geq 1$. For example, the percentile interval for θ in this case will be

$$[\hat{\theta}_{(1)}^*, \hat{\theta}_{(B)}^*],$$

where $\hat{\theta}_{(1)}^*$ and $\hat{\theta}_{(B)}^*$ are the smallest and largest bootstrap estimates of θ .

Consider the first approach in the cases of mean and variance. Let $\hat{\theta}_b^*$ be the $\hat{\theta}_b^*$ arranged in an increasing order. Then the intervals (2.7), (2.15) and (2.20) will be

$$\left[\hat{\theta}_{(1)}^* - \frac{\hat{\theta}_{(3)}^* - \hat{\theta}_{(1)}^*}{2/B}(-\hat{\alpha}), \quad \hat{\theta}_{(B)}^* + \frac{\hat{\theta}_{(B)}^* - \hat{\theta}_{(B-2)}^*}{2/B}(-\hat{\alpha}) \right].$$

In other words, the interval is extended at the two ends with the best histogram information from the sample, the histogram at the ends. The interval (2.9) for the mean will be changed to

$$\left[\hat{\theta} - \left(\hat{\theta}_{(B)}^* - \hat{\theta} + \frac{\hat{\theta}_{(B)}^* - \hat{\theta}_{(B-2)}^*}{2/B}(-\hat{\alpha}) \right), \quad \hat{\theta} - \left(\hat{\theta}_{(1)}^* - \hat{\theta} - \frac{\hat{\theta}_{(3)}^* - \hat{\theta}_{(1)}^*}{2/B}(-\hat{\alpha}) \right) \right].$$

The interval (2.10) for the variance will be changed to

$$\left[\hat{\theta} / \left(\frac{\hat{\theta}_{(B)}^*}{\hat{\theta}} + \frac{\hat{\theta}_{(B)}^* - \hat{\theta}_{(B-2)}^*}{2\hat{\theta}/B}(-\hat{\alpha}) \right), \quad \hat{\theta} / \left(\frac{\hat{\theta}_{(1)}^*}{\hat{\theta}} - \frac{\hat{\theta}_{(3)}^* - \hat{\theta}_{(1)}^*}{2\hat{\theta}/B}(-\hat{\alpha}) \right) \right].$$

But if $\left(\frac{\hat{\theta}_{(1)}^*}{\hat{\theta}} - \frac{\hat{\theta}_{(3)}^* - \hat{\theta}_{(1)}^*}{2\hat{\theta}/B}(-\hat{\alpha}) \right) \leq 0$, we use the following procedure.

Let $\hat{J}(y) = \#\{\log(\hat{\theta}_b^*/\hat{\theta}) \leq y\}/B$. Let a $100(1-2\hat{\alpha})\%$ confidence interval for $\log \theta$ be

$$[\log \hat{\theta} - \hat{J}^{-1}(1 - \hat{\alpha}), \quad \log \hat{\theta} - \hat{J}^{-1}(\hat{\alpha})].$$

Then the $100(1-2\hat{\alpha})\%$ confidence interval for $\log \theta$ is given by

$$\left[\log \hat{\theta} - \left(\log(\hat{\theta}_{(B)}^*/\hat{\theta}) + \frac{\log(\hat{\theta}_{(B)}^*/\hat{\theta}) - \log(\hat{\theta}_{(B-2)}^*/\hat{\theta})}{2/B}(-\hat{\alpha}) \right), \right. \\ \left. \log \hat{\theta} - \left(\log(\hat{\theta}_{(1)}^*/\hat{\theta}) - \frac{\log(\hat{\theta}_{(3)}^*/\hat{\theta}) - \log(\hat{\theta}_{(1)}^*/\hat{\theta})}{2/B}(-\hat{\alpha}) \right) \right].$$

Consequently, the interval (2.10) is

$$\left[\frac{\hat{\theta}^2}{\hat{\theta}_{(B)}^*} \left(\frac{\hat{\theta}_{(B)}^*}{\hat{\theta}_{(B-2)}^*} \right)^{2\hat{\alpha}/B}, \quad \frac{\hat{\theta}^2}{\hat{\theta}_{(1)}^*} \left(\frac{\hat{\theta}_{(3)}^*}{\hat{\theta}_{(1)}^*} \right)^{-2\hat{\alpha}/B} \right].$$

For the percentile-t methods, the t_b is now arranged in an increasing order. The $100(1-2\hat{\alpha})\%$ confidence intervals for the mean and the variance are, respectively,

$$\left[\bar{x} - \left(t_{(B)} + \frac{t_{(B)} - t_{(B-2)}}{2/B}(-\hat{\alpha}) \right) s/\sqrt{n}, \quad \bar{x} - \left(t_{(1)} - \frac{t_{(3)} - t_{(1)}}{2/B}(-\hat{\alpha}) \right) s/\sqrt{n} \right]$$

and

$$\left[s^2 - \left(t_{(B)} + \frac{t_{(B)} - t_{(B-2)}}{2/B}(-\hat{\alpha}) \right) \tau/\sqrt{n}, \quad s^2 - \left(t_{(1)} - \frac{t_{(3)} - t_{(1)}}{2/B}(-\hat{\alpha}) \right) \tau/\sqrt{n} \right].$$

In order to assess the quality of the model (3.19), we use the following distributions, which are excluded in building (3.19).

11. $\sqrt{X} + X^2$; $X \sim U(0, 1)$

12. $\exp(-X) + X^2$; $X \sim N(0, 1)$

13. Beta(0.5, 2)

14. Beta(1, 2)

15. Beta(4, 4)

(List 2)

16. Pareto(3, 2)

17. Pareto(6, 2)

- 18. Extreme value
- 19. Logistic
- 20. Gamma(1/3, 3)

The choice of these ten distributions is again arbitrary. The three beta distributions are chosen because of the vast variety in the shapes of the beta family.

4.2 Mean

The sample sizes used for the simulation were $n = 10, 20$ and 40 . The two-sided 90% and 95% confidence levels are considered. The results are contained in Tables 4.1-4.6. In the tables, the centralized percentile method is denoted as C. Perc., the percentile method as Perc. and the percentile-t method as Perc.-t. The estimated coverage probabilities for the modified methods are in parentheses.

Table 4.2 shows a typical result from our simulation for the mean. The tabulated values are the estimated coverages of a 90% confidence interval for the mean based on 800 simulated samples from the above 20 listed distributions. The sample size (n) is 20 and the bootstrap replicates $B=1000$. The results are compatible with a similar simulation result in Loh and Wu(1987). The general performance cannot be considered satisfactory, though the performance is good for the normal and the uniform. Thus, we suspect that the normal or the uniform is the preferred distribution for the bootstrap confidence interval for the mean. An investigation of all the simulations reveals that the overall performance of the normal is better than the uniform. Table 4.7 summarizes this evidence. Then, from the definition (3.5) of a preferred distribution, the preferred distribution for the case of mean is the normal distribution. This fact is also supported by the developments on $\mathcal{F}$ in the previous chapter. The last

columns of Tables 4.1-4.6 give the ESKSD between the listed distribution $F(x)$ and the preferred distribution. They were estimated by 800 samples. Note that according to (3.24), the distance between a sample from the preferred distribution and the distribution itself is not 0.

The sample correlation coefficients between coverage discrepancy and ESKSD for the first 10 distributions are given in the last rows of Tables 4.1-4.6. If we can consider the 10 populations in List 1 as random samples from all of the possible future population distributions, then a sample correlation of 0.632, in absolute value, is enough to reject the null hypothesis of no correlation (against the two-sided alternative) at 0.05 level (0.872 at 0.001 level). Thus, some of the sample correlations above 0.90 must be real. We see no reason to expect for this high correlation except that the concept of preferred distribution is valid.

From the theory in the previous section and the high correlation, ESKSD should be helpful for constructing a better confidence interval in small sample bootstrap.

Table 4.8 shows the estimates of parameters in (3.19) for the mean. The absolute values of the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ become smaller as the sample sizes increase as expected.

A summary of the simulated results is given in Table 4.9. The tabulated values are the averages of absolute differences between $(1 - 2\alpha)$ and the coverages based on all 20 distributions in Lists 1 and 2. These coverage discrepancies in the modified interval with the restriction $(1 - 2\hat{\alpha}) \leq 1$ are given in parentheses, while the coverage discrepancies without the restriction are given without parentheses. We see that the modified intervals are better than the ordinary intervals. There are differences between the coverage discrepancies in the modified intervals with or without the restriction $(1 - 2\hat{\alpha}) \leq 1$. The differences decrease as n increases. They become negligible at sample of size $n = 20$. Because of the computational burden of the double bootstrap, we only performed the simulation with $n=10$ for the B-method

and the modified B-method with $(1 - 2\hat{\alpha}) \leq 1$. We see that the coverage probabilities of the B-method is quite good for the mean.

Figures 4.1-4.6 show the reduction of discrepancy between the coverage by the five ordinary methods (solid lines) and the modified methods with the first approach (dash lines). Each figure consists of five small figures about the percentile, BC, BC_a , percentile-t, and centralized percentile methods. The first 10 distributions in the x-axis are those described in List 1 and the last 10 distributions are those described in List 2. The horizontal line at 90% or 95% in the y-axis is the nominal level of the confidence interval. Note that since the value in the x-axis refers to a distribution, there is no meaning to the slopes of the lines. We can see that the improvements by using the modified methods turn out to be substantial for all methods. The improvements has been reduced at the sample size $n = 40$. A possible reason will be the sample size is enough large for constructing confidence interval for the mean by the ordinary methods. The first 10 distributions (List 1) which are placed in left-half of a small figure are used for establishing (3.19). From the figures, we can see that even if a distribution is not included in the list, the modified methods work for better coverages.

We recommend the percentile-t method and the percentile method for small sample bootstrap confidence intervals due to their simplicity and coverage probabilities as shown. A preferred method, as argued Hall(1986), Beran(1988), and Efron(1992), will be to bootstrap an pivotal statistic, i.e., t -like statistic (2.11) in Section 2.2, since the ordinary percentile-t interval outperformed the other ordinary intervals.

We have also found that the modified percentile-t interval was better than the ordinary percentile-t interval. The other modified intervals such as the centralized percentile, the percentile, the BC, and the BC_a intervals were also shown to be better than the ordinary percentile-t interval. Moreover, the other modified intervals have the similar coverage probability error as the modified percentile-t interval.

Table 4.1. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 10$, $1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	86.6(86.3)	86.9(87.3)	86.9(88.1)	86.9(89.7)	92.3(88.9)	0.183
2. U(-1,1)	85.0(89.7)	86.4(92.3)	87.7(93.6)	89.5(94.5)	94.2(94.5)	0.192
3. MN I	85.9(91.4)	84.0(91.6)	82.1(92.0)	79.9(92.3)	87.6(90.4)	0.207
4. MN II	85.8(89.5)	84.7(90.3)	84.5(91.0)	82.5(91.8)	87.9(89.0)	0.190
5. χ_4^2	81.1(85.1)	80.9(88.0)	80.4(89.1)	81.0(90.1)	88.2(89.5)	0.208
6. D. Exp.	86.9(89.0)	82.5(89.4)	80.8(89.7)	79.1(90.1)	86.1(87.3)	0.205
7. Exp.	81.0(89.4)	83.9(91.6)	83.5(91.6)	85.1(92.9)	90.5(92.3)	0.236
8. t_2	87.1(92.8)	82.5(92.6)	79.8(92.6)	75.5(92.8)	85.4(89.4)	0.228
9. LogN	73.1(84.0)	74.1(86.8)	75.4(87.1)	76.4(88.4)	86.1(87.3)	0.259
10. Weib.	65.5(82.7)	68.3(92.1)	69.5(92.0)	71.2(91.8)	83.2(89.0)	0.309
11. UN I	83.2(88.4)	84.9(91.8)	86.1(92.8)	87.7(94.0)	93.5(93.5)	0.189
12. UN II	74.5(85.0)	77.2(91.5)	77.8(92.0)	79.4(92.0)	87.3(90.6)	0.286
13. B(.5,2)	80.6(89.1)	82.0(92.5)	82.5(93.0)	83.9(93.8)	90.8(92.6)	0.240
14. B(1,2)	84.9(87.4)	86.3(89.7)	86.3(90.9)	87.5(92.5)	92.8(93.3)	0.195
15. B(4,4)	85.8(85.4)	86.8(87.1)	87.1(87.9)	87.4(89.6)	92.6(90.1)	0.180
16. P(3,2)	72.1(85.4)	73.8(89.2)	74.5(90.0)	76.1(90.9)	84.4(88.2)	0.267
17. P(6,2)	77.1(83.2)	78.8(86.6)	79.0(87.4)	79.1(89.5)	87.0(88.0)	0.248
18. Extre.	83.1(85.5)	82.6(87.3)	83.2(88.2)	83.6(89.7)	90.0(88.5)	0.197
19. Logis.	84.2(87.3)	82.9(88.6)	82.4(88.7)	82.0(89.1)	87.0(88.0)	0.189
20. Gam.	70.0(86.9)	72.2(93.9)	74.1(94.1)	74.9(94.2)	86.5(91.8)	0.297
correlation	-0.913	-0.933	-0.918	-0.770	-0.675	—

Table 4.2. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 20, 1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	87.3(86.6)	87.9(88.1)	88.1(88.4)	87.5(89.9)	90.0(90.0)	0.134
2. U(-1,1)	87.4(88.5)	87.1(90.6)	88.4(91.3)	89.2(94.0)	92.4(93.5)	0.148
3. MN I	88.0(91.8)	85.6(91.4)	84.6(90.6)	83.2(89.2)	86.5(87.7)	0.164
4. MN II	87.4(89.7)	86.8(90.4)	86.5(89.9)	85.6(90.3)	88.1(90.1)	0.141
5. χ_4^2	87.0(86.9)	87.1(89.0)	87.4(89.4)	87.1(91.1)	90.4(90.3)	0.175
6. D. Exp.	88.1(92.5)	86.1(91.4)	85.5(90.8)	84.0(90.8)	86.5(89.0)	0.159
7. Exp.	84.4(90.4)	85.9(92.6)	85.1(93.4)	85.0(93.8)	89.0(93.0)	0.209
8. t_2	90.1(93.8)	85.6(92.4)	83.1(92.3)	79.3(91.3)	84.5(88.0)	0.198
9. LogN.	76.9(86.6)	78.9(87.6)	80.1(88.2)	81.0(89.4)	85.9(88.6)	0.240
10. Weib.	72.0(88.0)	75.1(92.8)	76.4(92.1)	77.8(92.1)	84.4(90.3)	0.304
11. UN I	86.3(89.2)	87.4(90.9)	88.4(91.6)	88.9(93.3)	92.5(93.8)	0.147
12. UN II	79.3(91.4)	79.6(93.0)	80.1(93.3)	80.4(93.6)	86.4(92.8)	0.269
13. B(.5,2)	85.9(91.1)	87.0(93.0)	87.5(92.9)	89.1(93.6)	92.6(92.5)	0.213
14. B(1,2)	87.1(90.8)	87.5(91.9)	87.6(92.0)	88.2(94.1)	91.3(94.0)	0.156
15. B(4,4)	85.0(86.5)	85.0(87.9)	84.9(88.0)	85.1(89.1)	88.0(90.3)	0.133
16. P(3,2)	75.6(87.3)	78.1(89.5)	79.1(89.7)	80.4(91.0)	86.3(89.6)	0.253
17. P(6,2)	81.1(89.6)	82.0(91.1)	82.9(91.4)	83.0(91.9)	87.1(90.6)	0.227
18. Extre.	88.4(87.7)	88.0(90.0)	87.9(90.1)	87.7(90.3)	90.8(89.5)	0.158
19. Logis.	88.1(87.7)	87.6(88.7)	87.5(88.9)	86.4(90.0)	89.4(89.7)	0.141
20. Gam.	79.8(94.2)	81.7(96.4)	81.9(96.1)	83.1(96.0)	89.6(94.5)	0.286
Correlation	-0.880	-0.931	-0.940	-0.798	-0.626	—

Table 4.3. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 40, \quad 1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	90.8(87.7)	90.0(89.2)	90.0(90.3)	90.1(90.5)	91.1(90.1)	0.095
2. U(-1,1)	85.1(88.9)	86.1(91.0)	86.6(92.3)	87.0(93.3)	87.6(92.3)	0.120
3. MN I	89.0(91.9)	86.4(91.0)	86.3(90.8)	84.7(89.6)	86.3(89.9)	0.137
4. MN II	89.7(89.7)	89.1(91.1)	88.9(90.6)	88.4(91.3)	90.4(91.1)	0.111
5. χ_4^2	88.4(88.7)	87.7(90.0)	88.0(90.4)	87.9(90.9)	90.0(90.6)	0.148
6. D. Exp.	92.1(89.0)	90.9(88.5)	89.7(88.5)	89.5(88.1)	91.3(86.9)	0.130
7. Exp.	86.6(90.9)	86.1(91.0)	86.1(91.3)	86.6(91.4)	88.6(91.0)	0.186
8. t_2	91.8(93.4)	87.3(91.9)	85.3(90.0)	81.2(87.4)	84.9(87.4)	0.184
9. LogN.	82.7(85.3)	84.1(88.5)	84.7(89.4)	86.3(90.1)	88.7(88.7)	0.229
10. Weib.	78.9(86.3)	81.0(88.4)	81.2(88.6)	82.4(89.6)	86.8(89.2)	0.304
11. UN I	88.7(89.6)	88.9(91.6)	89.0(91.6)	89.5(92.8)	90.4(92.8)	0.117
12. UN II	82.5(87.7)	84.1(90.3)	83.7(91.4)	83.7(92.4)	86.1(91.3)	0.261
13. B(.5,2)	87.1(91.1)	88.5(92.5)	89.4(93.1)	90.3(93.8)	90.8(94.0)	0.194
14. B(1,2)	90.9(90.6)	91.5(91.6)	92.4(91.9)	92.4(93.0)	93.5(92.4)	0.130
15. B(4,4)	88.6(88.7)	88.2(89.9)	88.4(89.9)	88.7(90.4)	91.0(90.4)	0.097
16. P(3,2)	80.1(86.3)	81.7(88.0)	82.4(89.0)	81.6(89.6)	86.0(88.9)	0.244
17. P(6,2)	85.1(90.8)	86.3(92.0)	86.1(92.0)	86.0(92.3)	88.4(91.6)	0.214
18. Extre.	89.2(88.9)	88.4(90.0)	88.7(89.9)	88.9(89.9)	90.5(90.4)	0.130
19. Logis.	88.9(88.9)	87.7(89.2)	87.7(89.4)	87.4(90.1)	88.7(90.1)	0.103
20. Gam.	85.3(93.5)	86.6(94.9)	87.3(95.4)	88.4(94.6)	91.1(93.5)	0.275
Correlation	-0.785	-0.877	-0.916	-0.690	-0.482	—

Table 4.4. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 10, \quad 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	91.1(90.3)	91.8(92.1)	91.4(92.8)	91.9(94.0)	95.6(94.0)	0.183
2. U(-1,1)	88.4(92.9)	91.3(94.6)	91.6(95.1)	93.1(96.5)	97.3(97.6)	0.192
3. MN I	92.1(94.6)	90.0(95.4)	89.2(95.9)	87.0(96.1)	92.5(94.6)	0.207
4. MN II	91.6(93.4)	89.6(94.5)	89.0(95.5)	88.2(96.4)	93.5(95.1)	0.190
5. χ_4^2	85.3(91.0)	86.0(93.0)	87.3(93.6)	86.6(94.9)	93.4(94.9)	0.208
6. D. Exp.	91.6(93.4)	89.2(94.6)	87.7(95.2)	85.6(96.0)	91.6(94.4)	0.205
7. Exp.	86.3(93.4)	87.9(95.6)	88.6(96.1)	89.6(96.7)	95.4(95.9)	0.236
8. t_2	91.9(95.5)	87.9(95.7)	86.6(95.6)	84.0(96.2)	92.6(95.4)	0.228
9. LogN.	77.0(88.7)	80.5(91.4)	81.4(92.1)	82.0(93.3)	92.5(92.5)	0.259
10. Weib.	68.3(88.0)	73.4(94.7)	74.6(94.7)	76.6(94.9)	91.1(93.8)	0.309
11. UN I	88.9(92.0)	91.0(95.7)	91.8(96.6)	92.9(97.5)	96.4(98.3)	0.189
12. UN II	77.4(90.8)	81.9(95.2)	83.1(95.2)	84.5(95.7)	92.3(95.2)	0.286
13. B(.5,2)	83.9(94.0)	86.1(96.4)	87.0(96.6)	87.6(96.7)	95.2(96.7)	0.240
14. B(1,2)	90.0(92.1)	91.1(94.5)	91.9(94.7)	92.3(95.5)	96.5(97.1)	0.195
15. B(4,4)	90.9(90.0)	91.4(92.3)	91.4(92.3)	91.8(94.0)	96.6(94.9)	0.180
16. P(3,2)	76.6(89.7)	79.8(94.0)	81.2(94.4)	81.4(95.2)	90.6(93.3)	0.267
17. P(6,2)	81.4(89.4)	83.1(94.2)	83.6(94.5)	84.9(95.0)	92.8(92.8)	0.248
18. Extre.	88.4(90.4)	88.7(92.9)	89.1(92.9)	89.7(94.2)	96.0(94.7)	0.197
19. Logis.	89.9(92.4)	88.6(93.3)	88.1(93.5)	87.4(94.6)	92.9(94.4)	0.189
20. Gam.	72.7(90.8)	76.9(96.7)	78.8(96.7)	81.1(96.9)	92.5(95.0)	0.297
Correlation	-0.899	-0.951	-0.949	-0.868	-0.907	—

Table 4.5. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 20, \quad 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0, 1)	93.0(91.5)	92.4(93.5)	92.3(93.6)	92.3(95.2)	94.4(95.7)	0.134
2. U(-1, 1)	92.0(93.8)	93.4(96.6)	94.1(97.0)	94.6(98.3)	97.5(98.0)	0.148
3. MN I	94.9(95.9)	92.0(96.1)	91.4(96.2)	89.4(97.0)	92.1(95.1)	0.164
4. MN II	92.9(93.0)	92.1(93.9)	91.6(94.6)	91.1(95.2)	93.4(94.1)	0.141
5. χ_4^2	90.6(91.8)	92.1(93.4)	91.8(93.6)	92.4(94.9)	94.6(94.5)	0.175
6. D. Exp.	93.0(96.4)	91.9(97.0)	90.6(97.1)	89.9(97.5)	92.4(94.7)	0.159
7. Exp.	88.9(96.5)	90.1(97.1)	89.5(97.3)	90.6(97.5)	94.1(96.6)	0.209
8. t_2	95.5(96.9)	92.1(96.9)	91.0(96.5)	86.5(97.1)	92.3(94.9)	0.198
9. LogN.	81.4(91.6)	84.5(93.3)	85.5(93.6)	86.0(94.4)	90.9(92.8)	0.240
10. Weib.	75.5(95.5)	81.1(97.8)	82.2(97.9)	84.0(97.8)	91.0(95.9)	0.304
11. UN I	91.8(94.4)	93.3(95.9)	93.3(96.1)	93.8(97.5)	96.6(97.4)	0.147
12. UN II	81.6(95.7)	85.4(97.1)	85.9(97.0)	86.6(97.5)	92.9(95.6)	0.269
13. B(.5,2)	91.0(96.2)	93.0(97.3)	93.4(97.4)	93.8(97.8)	96.9(96.6)	0.213
14. B(1,2)	91.6(95.1)	93.0(96.7)	93.5(97.0)	93.8(97.4)	95.7(97.5)	0.156
15. B(4,4)	89.9(90.9)	90.9(92.8)	91.8(93.6)	91.9(95.1)	94.1(94.7)	0.133
16. P(3,2)	79.6(93.6)	83.5(94.4)	84.5(94.5)	86.8(95.2)	92.6(94.4)	0.253
17. P(6,2)	84.9(95.5)	87.0(96.4)	87.3(96.4)	88.2(96.9)	92.5(95.6)	0.227
18. Extre.	92.1(92.4)	93.1(94.4)	92.6(94.5)	92.9(94.9)	95.0(94.1)	0.158
19. Logis.	94.1(93.3)	93.9(94.6)	93.5(95.4)	92.0(96.0)	95.0(95.4)	0.141
20. Gam.	83.2(97.5)	86.9(98.0)	87.7(97.9)	88.9(97.9)	94.4(97.8)	0.286
Correlation	-0.885	-0.932	-0.941	-0.835	-0.891	—

Table 4.6. Coverage probabilities (in %) of the bootstrap interval for the mean:
 $n = 40, 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0, 1)	94.7(92.8)	95.0(93.8)	94.9(93.8)	94.7(94.6)	95.6(94.4)	0.095
2. U(-1, 1)	91.4(94.9)	92.1(95.7)	92.5(96.5)	92.8(97.0)	94.5(96.5)	0.120
3. MN I	94.7(97.4)	92.9(96.9)	92.1(96.1)	91.4(95.5)	92.8(95.0)	0.137
4. MN II	94.6(94.2)	94.4(94.6)	94.1(94.6)	93.9(94.9)	95.1(94.9)	0.111
5. χ_4^2	92.4(94.5)	93.3(95.0)	93.5(95.0)	93.8(96.2)	95.0(96.0)	0.148
6. D. Exp.	95.5(94.7)	94.9(94.6)	95.0(94.6)	94.1(94.7)	95.0(94.0)	0.130
7. Exp.	90.8(95.5)	91.9(96.1)	92.3(96.1)	92.5(96.5)	93.9(95.2)	0.186
8. t_2	96.4(96.9)	93.0(96.0)	91.3(95.2)	88.1(95.0)	91.1(93.3)	0.184
9. LogN.	87.5(94.1)	90.5(94.9)	90.6(94.9)	90.9(95.6)	93.9(94.5)	0.229
10. Weib.	81.7(95.6)	85.0(97.5)	86.0(97.3)	87.9(96.6)	92.0(95.4)	0.304
11. UN I	93.3(95.2)	93.8(96.0)	93.8(96.6)	94.2(97.1)	95.2(97.4)	0.117
12. UN II	85.5(95.9)	88.0(96.5)	88.9(96.1)	89.2(96.2)	92.4(95.2)	0.261
13. B(.5,2)	92.0(98.4)	92.9(98.8)	93.1(98.4)	93.8(98.4)	95.6(97.8)	0.194
14. B(1,2)	95.1(95.6)	95.4(96.0)	95.6(96.1)	95.9(96.9)	97.5(96.9)	0.130
15. B(4,4)	94.1(92.4)	94.6(93.6)	94.2(94.2)	94.5(95.0)	95.7(94.7)	0.097
16. P(3,2)	84.5(93.4)	88.2(95.6)	88.4(95.6)	88.9(96.2)	92.5(95.5)	0.244
17. P(6,2)	89.1(95.5)	91.3(96.4)	91.5(96.6)	92.3(96.7)	94.4(95.5)	0.214
18. Extre.	94.6(94.0)	94.2(94.4)	94.2(94.4)	94.2(94.6)	95.9(95.2)	0.130
19. Logis.	93.9(93.4)	93.1(93.5)	93.4(93.3)	93.1(93.9)	95.0(94.5)	0.103
20. Gam.	89.0(98.5)	91.1(99.1)	91.9(99.1)	93.1(98.9)	95.9(97.8)	0.275
Correlation	-0.846	-0.923	-0.931	-0.807	-0.860	—

Table 4.7. Summary of bootstrap mean confidence interval coverage probabilities (in %) for the normal(N) and uniform(U) distributions

Sample Size	$n=10$				$n=20$				$n=40$			
	90%		95%		90%		95%		90%		95%	
	N	U	N	U	N	U	N	U	N	U	N	U
C. Perc.	87	85	91	88	87	87	93	92	91	85	95	91
Perc.	87	86	92	91	88	87	92	93	90	86	95	92
BC	87	88	91	92	88	88	92	94	90	87	95	93
BC_a	87	90	92	93	88	89	92	95	90	87	95	93
Perc.-t	92	94	96	97	90	92	94	98	91	88	96	95

Table 4.8. Regression coefficients in (3.19) for the mean

Sample size	Method	90%		95%	
		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
$n = 10$	C. Perc.	-0.296	1.704	-0.324	1.853
	Perc.	-0.229	1.422	-0.230	1.410
	BC	-0.199	1.300	-0.199	1.270
	BC_a	-0.154	1.113	-0.157	1.095
	Perc.-t	-0.111	0.586	-0.047	0.277
$n = 20$	C. Perc.	-0.129	0.964	-0.149	1.076
	Perc.	-0.084	0.737	-0.085	0.714
	BC	-0.071	0.674	-0.068	0.628
	BC_a	-0.045	0.563	-0.044	0.521
	Perc.-t	-0.037	0.316	-0.026	0.232
$n = 40$	C. Perc.	-0.061	0.524	-0.068	0.597
	Perc.	-0.034	0.399	-0.043	0.424
	BC	-0.029	0.381	-0.036	0.386
	BC_a	-0.016	0.315	-0.021	0.309
	Perc.-t	-0.013	0.164	-0.014	0.153

Table 4.9. Mean coverage discrepancy in absolute value (in %) by ordinary and modified methods (based on all 20 distributions in Lists 1 and 2)

Method	$n=10$		$n=20$		$n=40$	
	90%	95%	90%	95%	90%	95%
C. Perc.	9.3	9.8	5.8	6.2	3.5	3.7
Mod. C. Perc.	3.3(6.3)	3.4(7.9)	2.1(2.5)	1.7(3.5)	1.8(1.9)	1.3(1.7)
Perc.	8.9	8.7	5.5	4.9	3.2	2.8
Mod. Perc.	2.0(2.8)	1.2(3.7)	1.9(1.7)	1.6(1.5)	1.4(1.4)	1.3(1.1)
BC	8.8	8.3	5.4	4.8	3.2	2.7
Mod. BC	2.0(2.6)	1.2(3.5)	1.8(1.6)	1.6(1.4)	1.3(1.3)	1.3(1.1)
BC_a	8.6	8.1	5.4	4.7	3.2	2.6
Mod. BC_a	1.9(2.3)	1.0(3.1)	2.0(1.8)	1.6(1.5)	1.6(1.6)	1.3(1.1)
Perc.-t	3.0	2.0	2.4	1.8	1.9	1.2
Mod. Perc.-t	1.8(1.8)	1.1(1.1)	1.7(1.7)	1.2(1.2)	1.5(1.5)	1.0(1.0)
$B_{n,1}$ from S_n	6.1	-	-	-	-	-
Mod. $B_{n,1}$ from S_n	(3.4)	-	-	-	-	-
$B_{n,1}$ from T_n	2.5	-	-	-	-	-
Mod. $B_{n,1}$ from T_n	(1.9)	-	-	-	-	-

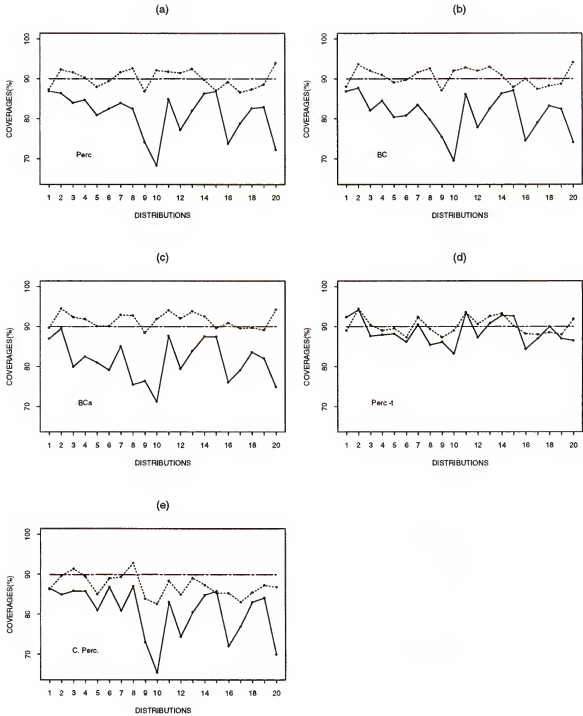


Figure 4.1. Coverage probabilities (in %) of a 90% confidence interval for μ ($n=10$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

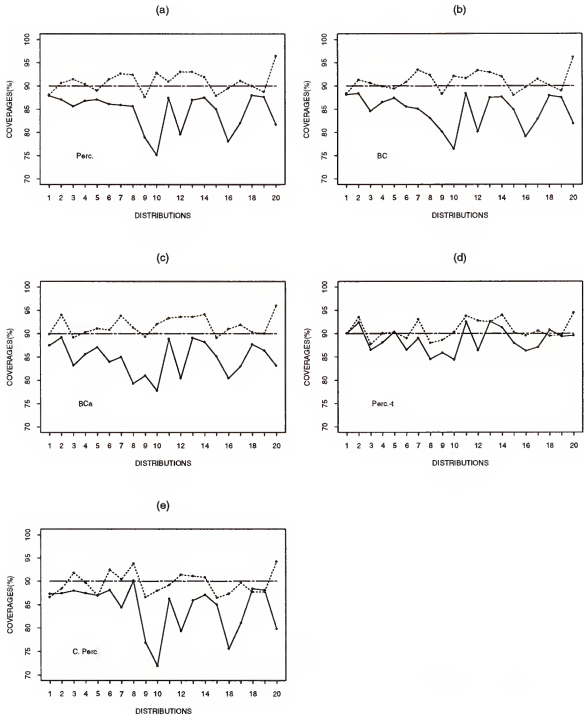


Figure 4.2. Coverage probabilities (in %) of a 90% confidence interval for μ ($n=20$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

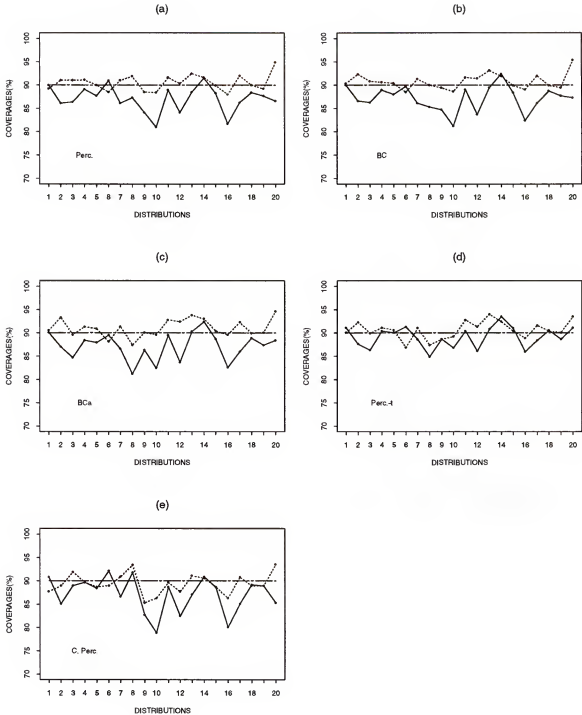


Figure 4.3. Coverage probabilities (in %) of a 90% confidence interval for μ ($n=40$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

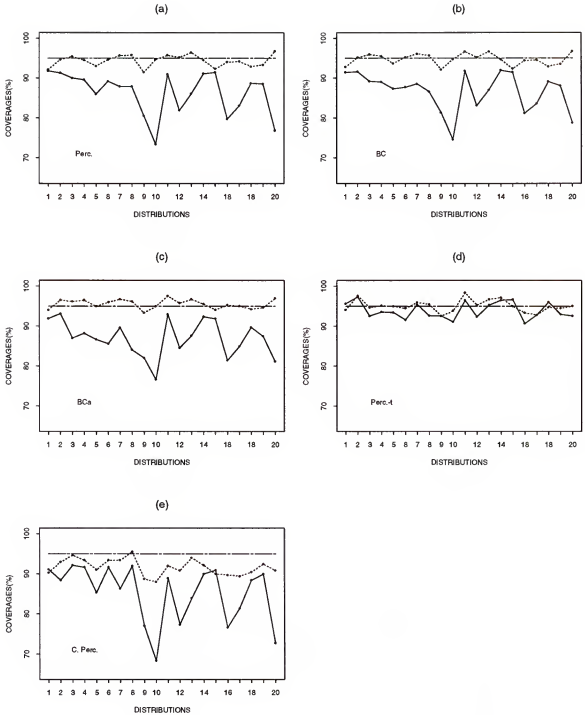


Figure 4.4. Coverage probabilities (in %) of a 95% confidence interval for μ ($n=10$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

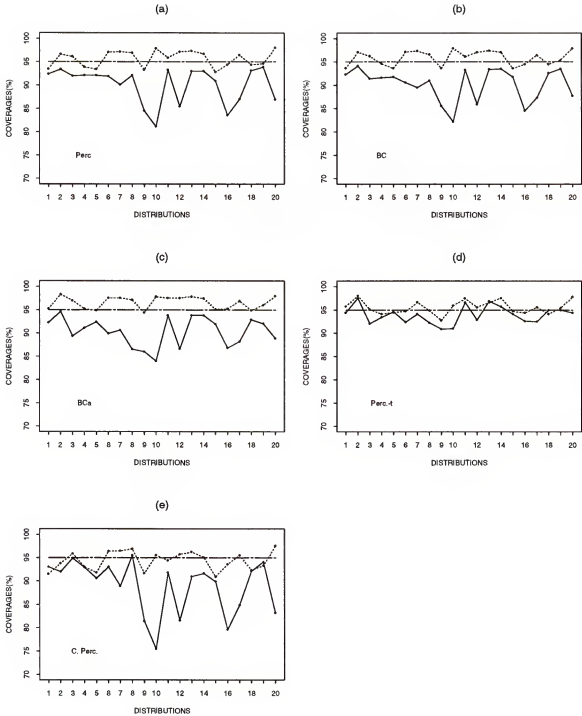


Figure 4.5. Coverage probabilities (in %) of a 95% confidence interval for μ ($n=20$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

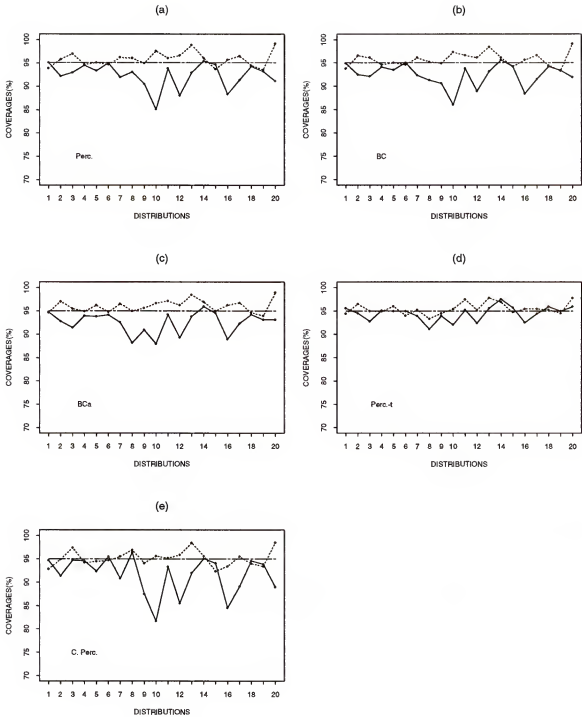


Figure 4.6. Coverage probabilities (in %) of a 95% confidence interval for μ ($n=40$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

4.3 Variance

The details of simulation are same as the mean. The sample sizes used for the simulation were $n = 10, 20$ and 40 . The two-sided 90% and 95% confidence levels are considered. The results are contained in Tables 4.10-4.15.

We suspect that the normal or the uniform is the preferred distribution for the bootstrap confidence interval for the variance. An investigation of all the simulations reveals that the overall performance of the uniform is better than the normal. Table 4.16 summarizes this evidence.

The sample correlation coefficients between coverage discrepancy and ESKSD for the first 10 distributions are given in the last rows of Tables 4.10-4.15. The smallest sample correlation for all cases is greater than 0.90 in absolute value. The range of the sample correlation coefficients is from 0.908 to 0.959. The null hypothesis of no correlation (against the two-sided alternative) at 0.001 level is rejected for all cases considered. We again see there is no reason to expect for this high correlation except that the concept of preferred distribution is valid.

Table 4.17 shows the estimates of parameters in (3.19) for variance. Again, the absolute values of the estimates for every method become smaller as the sample sizes increase.

A summary of the simulated results is given in Table 4.18. The tabulated values are the averages of absolute differences between $(1 - 2\alpha)$ and the coverages based on all 20 distributions in Lists 1 and 2. As Schenker(1985) pointed out, the centralized percentile method is promising, when its coverage probability error is compared with the percentile or BC methods. The BC_a method, which was intended to improve the BC method, has better coverage than the BC and percentile methods. However, the percentile-t method outperforms the BC_a methods. Also, it was found that the

modified methods outperforms the ordinary methods. There are differences between the coverage discrepancies in the modified intervals with or without the restriction $(1 - 2\hat{\alpha}) \leq 1$. The differences for the centralized percentile and percentile-t methods decrease as n increases. They become negligible at sample of size $n = 20$. As in the mean case, we only performed the simulation with $n=10$ for the B-method and the modified B-method with $(1 - 2\hat{\alpha}) \leq 1$. The coverage probabilities of the B-method were quite poor.

Figures 4.7-4.12 show the reduction of discrepancy between the coverage by the five ordinary methods (solid lines) and the modified methods with the first approach (dash lines). Each figure consists of five small figures about the percentile, BC, BC_a , percentile-t, and centralized percentile methods. The first 10 distributions in the x-axis are those described in List 1 and the last 10 distributions are those described in List 2. The horizontal line at 90% or 95% in the y-axis is the nominal level of the confidence interval.

We can see that the improvements by using the modified methods turn out to be substantial for all methods. The first 10 distributions (List 1) which are placed in left-half of a small figure were used for establishing (3.19). As in the case of the mean, even if a distribution was not included in the list, the modified methods worked for better coverages.

The percentile-t method and the centralized percentile method can be suggested for small sample bootstrap confidence intervals for the variance, due to their simplicity and coverage probabilities as shown. As in the case of the mean, a preferred procedure would be to bootstrap an t -like statistic, since the ordinary percentile-t interval outperformed the other ordinary intervals.

It was also found that the modified percentile-t interval was better than the ordinary percentile-t interval. The other modified intervals of the centralized percentile

interval, percentile interval, BC interval, and BC_a interval were also shown to be better than the ordinary percentile-t interval. Moreover, the other modified intervals for the sample size 20 or 40 have the similar coverage probability error as the modified percentile-t interval. This fact also can be found from Figures 4.7-4.12. We can see that all modified intervals have similar shapes in the figures.

Table 4.10. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 10, \quad 1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	82.1(90.0)	73.9(88.2)	75.9(87.3)	76.6(86.1)	88.1(87.7)	0.186
2. U(-1,1)	83.2(91.1)	83.5(90.4)	82.6(91.4)	84.2(90.9)	88.4(88.9)	0.179
3. MN I	69.6(91.0)	57.0(79.8)	58.9(80.8)	60.1(80.3)	81.0(85.9)	0.218
4. MN II	76.6(92.4)	70.4(90.4)	70.5(89.7)	71.2(89.2)	85.1(90.0)	0.201
5. χ_4^2	73.4(92.4)	61.6(81.1)	63.7(82.6)	64.5(83.0)	82.1(84.9)	0.209
6. D. Exp.	75.4(92.6)	65.1(85.5)	69.2(84.5)	70.6(84.7)	86.0(88.6)	0.218
7. Exp.	67.4(92.0)	57.0(84.0)	58.5(83.5)	59.5(83.0)	81.1(85.6)	0.236
8. t_4	74.9(91.4)	62.6(81.4)	66.1(81.5)	68.6(80.8)	83.4(85.4)	0.209
9. LogN.	56.9(89.6)	36.0(80.0)	38.4(79.5)	40.1(78.5)	68.9(81.6)	0.259
10. Weib.	56.9(96.5)	31.9(76.5)	33.6(75.5)	35.6(74.9)	69.9(92.5)	0.312
11. UN I	81.4(88.9)	82.4(90.4)	80.6(90.3)	81.9(88.9)	87.3(87.7)	0.179
12. UN II	60.6(94.1)	38.0(79.6)	40.4(79.0)	41.4(78.8)	74.0(88.5)	0.288
13. B(.5,2)	72.0(95.0)	65.6(90.4)	66.1(89.5)	67.0(89.2)	83.5(90.5)	0.240
14. B(1,2)	78.3(89.7)	75.1(89.4)	74.1(87.4)	74.9(87.4)	86.3(87.0)	0.192
15. B(4,4)	80.1(88.6)	80.4(87.3)	78.1(84.9)	77.8(84.4)	86.1(86.4)	0.179
16. P(3,2)	53.3(90.3)	27.6(72.2)	29.0(71.9)	31.4(71.0)	64.1(79.1)	0.267
17. P(6,2)	61.9(90.8)	44.5(79.5)	48.0(78.6)	50.2(77.5)	72.9(81.7)	0.249
18. Extre.	76.2(90.5)	62.1(84.0)	68.6(83.1)	68.4(82.5)	85.6(87.1)	0.198
19. Logis.	81.5(91.8)	70.9(84.7)	73.1(85.8)	73.2(86.1)	88.5(89.7)	0.198
20. Gam.	62.1(97.9)	46.1(86.0)	47.0(85.0)	48.0(84.9)	74.9(95.0)	0.299
Correlation	-0.933	-0.945	-0.950	-0.947	-0.908	—

Table 4.11. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 20, \quad 1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0, 1)	84.6(91.9)	81.7(90.4)	83.6(90.3)	84.0(89.9)	89.5(90.4)	0.148
2. U(-1, 1)	87.4(90.1)	89.1(94.5)	88.0(92.9)	88.6(93.1)	90.4(90.0)	0.129
3. MN I	73.8(91.5)	70.9(87.5)	72.0(86.9)	73.2(86.4)	82.2(88.5)	0.195
4. MN II	83.0(92.8)	78.9(92.0)	81.1(90.8)	81.6(90.3)	88.4(90.1)	0.167
5. χ_4^2	75.4(91.0)	69.1(84.7)	71.4(84.7)	73.4(85.5)	83.0(87.3)	0.183
6. D. Exp.	76.5(94.1)	70.2(91.3)	71.9(90.4)	72.0(90.4)	82.9(91.6)	0.187
7. Exp.	71.4(95.4)	63.5(91.9)	66.1(91.5)	67.6(91.8)	80.9(91.5)	0.220
8. t_4	78.4(91.1)	67.4(86.0)	73.0(85.0)	74.5(85.3)	83.6(86.9)	0.134
9. LogN.	57.0(94.7)	46.2(87.3)	48.1(86.4)	51.1(85.8)	68.8(88.5)	0.254
10. Weib.	56.5(99.5)	44.6(90.9)	45.9(90.5)	49.3(89.6)	70.4(98.9)	0.326
11. UN I	84.9(88.1)	85.8(92.8)	85.3(89.1)	86.3(89.4)	89.2(88.1)	0.134
12. UN II	59.5(98.4)	48.5(89.7)	49.8(89.2)	51.9(88.1)	72.5(93.1)	0.293
13. B(.5,2)	79.8(97.8)	79.4(96.9)	78.6(96.6)	79.4(96.6)	87.9(96.4)	0.231
14. B(1,2)	80.8(91.0)	81.4(91.9)	81.4(91.5)	80.9(92.0)	86.5(90.0)	0.152
15. B(4,4)	86.6(86.5)	82.4(90.9)	84.6(87.7)	85.5(87.1)	90.4(87.1)	0.139
16. P(3,2)	53.1(95.2)	37.0(83.5)	38.0(82.0)	42.0(81.0)	64.9(86.4)	0.270
17. P(6,2)	64.8(93.6)	54.6(90.3)	57.4(89.1)	58.7(88.4)	76.9(89.1)	0.245
18. Extre.	78.0(90.4)	72.9(86.3)	75.2(85.0)	76.9(86.1)	85.4(87.9)	0.170
19. Logis.	84.2(91.5)	76.9(88.7)	81.1(88.7)	81.9(88.7)	89.4(89.2)	0.164
20. Gam.	69.9(99.4)	58.1(95.9)	59.4(95.6)	61.5(95.2)	79.4(98.4)	0.311
Correlation	-0.951	-0.943	-0.957	-0.957	-0.925	—

Table 4.12. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 40, \quad 1 - 2\alpha = 0.90$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	86.5(90.4)	82.9(90.3)	85.9(89.2)	85.6(88.9)	89.6(88.4)	0.119
2. U(-1,1)	91.5(89.2)	89.7(90.8)	91.3(89.1)	92.3(90.0)	92.5(88.7)	0.096
3. MN I	79.8(92.6)	77.1(90.5)	79.4(89.0)	79.4(90.5)	86.6(91.0)	0.177
4. MN II	85.3(93.4)	82.7(93.9)	83.9(92.6)	85.8(92.5)	89.2(92.1)	0.145
5. χ_4^2	80.1(90.0)	77.1(87.0)	79.3(86.8)	80.1(87.3)	85.8(87.3)	0.161
6. D. Exp.	83.9(91.8)	82.1(90.6)	84.5(90.1)	84.4(91.0)	88.5(89.4)	0.167
7. Exp.	76.2(95.2)	73.1(94.9)	74.8(93.3)	76.7(92.1)	84.1(91.4)	0.212
8. t_4	78.8(89.4)	72.1(83.7)	75.5(83.7)	78.0(84.7)	84.2(86.3)	0.158
9. LogN.	63.5(93.4)	54.8(89.9)	57.9(87.9)	61.0(85.9)	75.6(84.1)	0.259
10. Weib.	65.0(99.5)	55.9(95.4)	57.7(95.0)	61.8(94.5)	75.9(97.8)	0.343
11. UN I	87.4(88.7)	88.5(93.4)	89.4(89.5)	90.0(89.4)	90.9(89.5)	0.097
12. UN II	68.4(99.4)	60.5(97.3)	62.6(96.2)	65.4(95.2)	79.5(95.9)	0.308
13. B(.5,2)	84.7(99.1)	82.0(99.3)	83.6(99.0)	84.9(98.5)	89.7(95.5)	0.227
14. B(1,2)	86.4(90.9)	87.0(92.9)	86.6(91.6)	86.3(91.6)	90.0(90.4)	0.125
15. B(4,4)	88.6(90.9)	86.3(92.0)	87.1(89.7)	88.1(89.9)	89.9(90.5)	0.107
16. P(3,2)	58.9(96.0)	42.5(89.7)	44.9(87.4)	51.9(85.0)	67.9(83.9)	0.280
17. P(6,2)	69.4(95.4)	62.7(93.9)	66.1(91.8)	68.3(90.1)	80.9(89.1)	0.245
18. Extre.	81.9(90.1)	77.4(86.6)	81.2(87.5)	81.7(87.3)	86.1(87.6)	0.146
19. Logis.	84.9(89.9)	82.4(89.1)	83.7(89.2)	84.6(88.5)	88.6(88.4)	0.136
20. Gam.	74.1(100.)	69.1(98.9)	71.4(98.9)	72.9(98.8)	83.7(98.3)	0.318
Correlation	-0.931	-0.918	-0.934	-0.932	-0.927	—

Table 4.13. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 10, \quad 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	87.5(94.4)	77.5(91.9)	81.1(90.0)	82.4(89.0)	94.7(91.8)	0.186
2. U(-1,1)	87.1(96.4)	88.1(93.6)	88.6(93.0)	89.7(92.9)	94.6(93.5)	0.179
3. MN I	78.0(95.2)	60.5(85.6)	63.5(84.0)	65.4(82.9)	89.4(88.1)	0.218
4. MN II	82.1(96.0)	74.6(93.8)	76.5(92.3)	78.3(91.1)	92.6(92.6)	0.201
5. χ_4^2	81.7(96.1)	64.4(86.8)	68.8(85.4)	69.7(85.4)	90.3(90.4)	0.209
6. D. Exp.	81.7(96.4)	70.1(88.9)	74.1(87.7)	75.6(87.4)	91.9(90.9)	0.218
7. Exp.	75.6(95.9)	59.7(89.1)	62.7(87.3)	64.9(86.1)	88.6(89.0)	0.236
8. t_4	82.5(95.5)	66.9(86.3)	70.7(85.4)	72.5(84.5)	91.5(89.9)	0.209
9. LogN.	65.4(93.9)	39.1(82.5)	41.1(80.9)	43.8(80.5)	78.0(84.9)	0.259
10. Weib.	64.9(98.1)	34.0(78.8)	36.0(78.0)	38.0(77.5)	79.3(93.5)	0.311
11. UN I	87.1(94.6)	86.4(94.4)	87.7(92.5)	88.9(92.3)	94.4(91.0)	0.179
12. UN II	66.9(96.2)	40.2(82.5)	42.4(81.9)	44.9(80.5)	81.7(90.8)	0.288
13. B(.5,2)	77.4(97.6)	69.9(92.9)	70.5(92.6)	72.2(91.5)	90.4(93.1)	0.240
14. B(1,2)	84.2(94.0)	80.1(92.6)	80.8(90.9)	82.1(89.7)	93.3(91.5)	0.192
15. B(4,4)	85.8(93.0)	84.4(91.3)	84.9(88.1)	85.9(87.5)	94.4(91.3)	0.179
16. P(3,2)	62.0(94.6)	29.4(75.5)	31.2(73.5)	33.7(73.0)	72.2(83.6)	0.267
17. P(6,2)	68.8(95.6)	47.9(82.6)	51.6(81.1)	53.3(80.4)	80.8(85.5)	0.249
18. Extre.	83.7(94.7)	67.5(87.9)	72.2(85.8)	74.0(85.8)	92.4(89.9)	0.198
19. Logis.	86.9(96.0)	74.6(89.6)	77.6(88.7)	78.5(87.6)	94.1(92.1)	0.198
20. Gam.	70.6(98.8)	47.9(87.6)	50.2(87.3)	51.6(86.3)	84.5(95.9)	0.299
Correlation	-0.944	-0.946	-0.952	-0.953	-0.915	—

Table 4.14. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 20, \quad 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	90.9(94.7)	86.8(94.7)	90.1(94.1)	91.1(93.1)	95.5(94.0)	0.148
2. U(-1,1)	92.0(94.9)	93.0(97.5)	93.1(96.5)	93.6(96.2)	96.2(94.7)	0.129
3. MN I	81.4(95.0)	74.5(92.0)	77.0(90.6)	79.0(90.0)	89.2(91.6)	0.195
4. MN II	88.7(96.1)	84.0(95.1)	85.9(94.1)	87.5(94.0)	93.1(94.5)	0.167
5. χ_4^2	82.7(95.1)	74.4(90.0)	78.3(88.6)	80.1(88.1)	90.6(90.8)	0.183
6. D. Exp.	83.4(97.9)	74.6(95.4)	77.5(94.9)	79.1(94.6)	90.3(95.1)	0.187
7. Exp.	78.1(97.1)	68.5(95.9)	70.4(94.4)	74.9(94.0)	87.0(94.6)	0.220
8. t_4	84.9(94.6)	73.0(91.5)	77.5(89.5)	79.1(89.2)	91.1(90.6)	0.177
9. LogN.	64.1(97.3)	48.5(90.8)	51.9(89.4)	54.9(87.9)	77.5(92.4)	0.254
10. Weib.	64.8(99.6)	46.6(92.0)	49.5(91.8)	52.9(90.9)	78.6(99.3)	0.326
11. UN I	89.2(92.3)	90.9(96.7)	91.6(94.4)	92.3(92.9)	94.4(92.0)	0.134
12. UN II	67.3(99.4)	50.5(92.6)	53.0(91.5)	56.6(90.1)	80.5(95.9)	0.293
13. B(.5,2)	86.1(98.9)	81.6(97.8)	83.7(97.5)	84.7(97.4)	93.9(97.8)	0.231
14. B(1,2)	86.6(94.9)	86.9(94.9)	86.9(94.2)	86.8(93.5)	92.4(93.9)	0.152
15. B(4,4)	90.9(92.1)	87.1(94.4)	89.5(92.1)	90.0(90.9)	94.4(91.3)	0.139
16. P(3,2)	61.0(97.5)	38.4(86.9)	41.6(85.5)	46.0(84.0)	73.4(91.0)	0.270
17. P(6,2)	72.1(96.9)	58.2(93.1)	60.9(91.8)	63.5(91.0)	83.7(91.6)	0.245
18. Extre.	84.5(94.9)	77.6(91.3)	82.0(89.7)	82.2(89.2)	91.0(91.4)	0.170
19. Logis.	91.1(96.0)	83.1(92.5)	86.4(91.6)	88.5(92.0)	95.9(93.1)	0.164
20. Gam.	76.1(99.5)	61.6(96.9)	64.0(96.4)	65.6(95.9)	86.6(99.1)	0.311
Correlation	-0.945	-0.951	-0.959	-0.955	-0.938	—

Table 4.15. Coverage probabilities (in %) of the bootstrap interval for the variance:
 $n = 40, \quad 1 - 2\alpha = 0.95$

Distribution	C. Perc.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. N(0,1)	92.4(95.0)	89.9(96.1)	91.1(94.4)	91.8(93.6)	94.7(94.5)	0.119
2. U(-1,1)	94.9(93.9)	94.7(94.5)	95.9(93.5)	95.6(94.2)	96.7(93.9)	0.096
3. MN I	85.8(96.1)	82.4(95.9)	84.2(94.7)	86.5(94.1)	92.3(94.9)	0.177
4. MN II	91.9(97.0)	87.9(97.4)	90.0(97.0)	90.8(96.7)	94.5(96.4)	0.145
5. χ_4^2	86.8(94.6)	83.0(92.6)	85.0(91.4)	85.4(91.5)	92.0(91.9)	0.161
6. D. Exp.	90.0(96.6)	87.5(96.1)	88.5(94.4)	89.9(94.5)	93.3(94.2)	0.167
7. Exp.	83.2(98.0)	78.1(97.3)	80.0(96.6)	82.7(96.2)	89.5(95.2)	0.212
8. t_4	86.1(94.4)	77.6(91.5)	81.6(90.1)	84.2(90.6)	90.4(90.9)	0.158
9. LogN.	69.5(97.4)	59.0(93.3)	62.0(92.0)	67.5(90.4)	82.2(91.4)	0.259
10. Weib.	72.2(99.5)	58.6(96.6)	61.5(96.1)	66.4(95.1)	82.1(98.8)	0.343
11. UN I	94.0(93.9)	93.8(96.0)	93.6(94.1)	94.7(95.0)	95.9(95.1)	0.097
12. UN II	75.0(99.5)	63.6(98.1)	66.4(97.4)	70.5(96.9)	84.5(99.1)	0.308
13. B(.5,2)	89.7(99.5)	87.5(99.3)	89.0(99.3)	89.5(99.3)	94.6(98.8)	0.227
14. B(1,2)	91.0(96.4)	91.5(95.1)	92.4(94.6)	93.0(95.5)	94.6(95.2)	0.125
15. B(4,4)	92.1(94.6)	91.5(95.6)	92.4(93.6)	93.0(94.0)	94.2(94.7)	0.107
16. P(3,2)	66.8(98.8)	45.9(92.0)	50.4(90.6)	57.2(89.7)	76.4(93.4)	0.280
17. P(6,2)	76.7(98.5)	67.3(96.4)	69.1(95.7)	74.6(94.6)	86.0(94.4)	0.245
18. Extre.	87.6(96.2)	83.9(93.9)	86.5(92.4)	87.1(93.0)	92.1(93.1)	0.146
19. Logis.	91.6(95.2)	87.4(95.0)	89.7(94.0)	90.5(94.1)	93.9(94.1)	0.136
20. Gam.	82.7(100.)	73.4(99.0)	76.0(98.9)	77.9(98.9)	90.6(99.5)	0.318
Correlation	-0.921	-0.933	-0.944	-0.945	-0.942	—

Table 4.16. Summary of bootstrap variance confidence interval coverage probabilities (in %) for the normal(N) and uniform(U) distributions

Sample Size	$n=10$				$n=20$				$n=40$			
	90%		95%		90%		95%		90%		95%	
	N	U	N	U	N	U	N	U	N	U	N	U
C. Perc.	82	83	88	87	85	87	91	92	87	92	92	95
Perc.	74	84	78	88	82	89	87	93	83	90	90	95
BC	76	83	81	89	84	88	90	93	86	91	91	96
BC_a	77	84	82	90	84	89	91	94	86	92	92	96
Perc.-t	88	88	95	95	90	90	96	96	90	93	95	97

Table 4.17. Regression coefficients in (3.19) for the variance

Sample size	Method	90%		95%	
		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
$n = 10$	C. Perc.	-0.306	2.199	-0.267	1.935
	Perc.	-0.559	3.864	-0.582	4.028
	BC	-0.558	3.775	-0.618	4.064
	BC_a	-0.558	3.713	-0.616	3.979
	Perc.-t	-0.270	1.601	-0.248	1.381
$n = 20$	C. Perc.	-0.194	1.763	-0.183	1.623
	Perc.	-0.248	2.350	-0.272	2.510
	BC	-0.264	2.332	-0.289	2.457
	BC_a	-0.246	2.167	-0.278	2.297
	Perc.-t	-0.157	1.192	-0.148	1.050
$n = 40$	C. Perc.	-0.103	1.158	-0.100	1.074
	Perc.	-0.117	1.465	-0.141	1.591
	BC	-0.138	1.460	-0.150	1.527
	BC_a	-0.126	1.310	-0.128	1.290
	Perc.-t	-0.085	0.723	-0.078	0.653

Table 4.18. Variance coverage discrepancy in absolute value (in %) by ordinary and modified methods (based on all 20 distributions in Lists 1 and 2)

Method	$n=10$		$n=20$		$n=40$	
	90%	95%	90%	95%	90%	95%
C. Perc.	18.8	17.0	15.7	14.2	11.4	10.0
Mod. C. Perc.	2.2(4.0)	1.2(6.1)	3.7(2.4)	1.9(3.8)	3.6(2.3)	2.1(2.2)
Perc.	30.4	31.8	22.1	23.0	15.7	15.8
Mod. Perc.	6.1(21.9)	7.3(25.5)	2.8(11.5)	2.4(14.5)	3.4(6.9)	1.8(8.3)
BC	28.9	29.4	20.4	20.5	13.8	13.8
Mod. BC	6.6(21.7)	8.7(25.8)	2.8(11.4)	3.1(14.6)	3.0(6.6)	2.1(8.6)
BC_a	27.7	27.7	19.0	18.6	12.3	11.6
Mod. BC_a	7.0(21.7)	9.4(25.9)	2.9(11.3)	3.7(14.7)	2.9(6.4)	2.0(8.3)
Perc.-t	9.1	6.6	8.0	6.5	5.4	4.7
Mod. Perc.-t	3.6(3.5)	4.6(4.8)	2.6(2.3)	2.5(2.5)	3.0(2.8)	1.8(1.7)
$B_{n,1}$ from U_n	32.9	-	-	-	-	-
Mod. $B_{n,1}$ from U_n	(21.3)	-	-	-	-	-

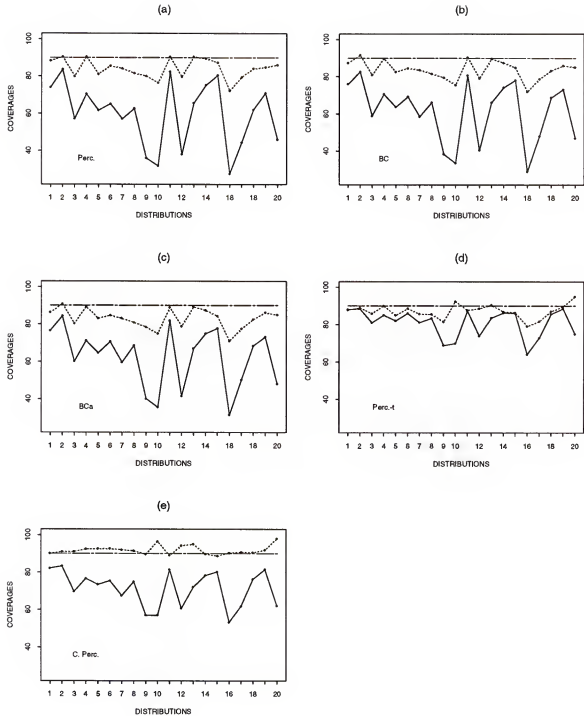


Figure 4.7. Coverage probabilities (in %) of a 90% confidence interval for σ^2 ($n=10$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

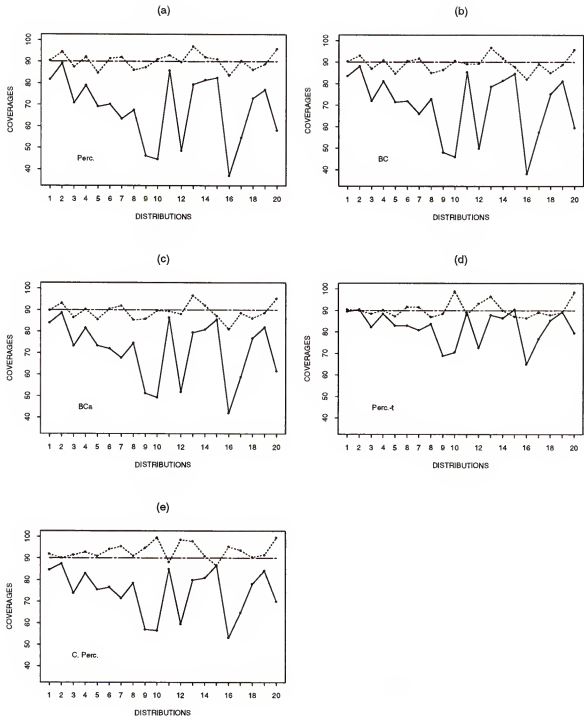


Figure 4.8. Coverage probabilities (in %) of a 90% confidence interval for σ^2 ($n=20$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

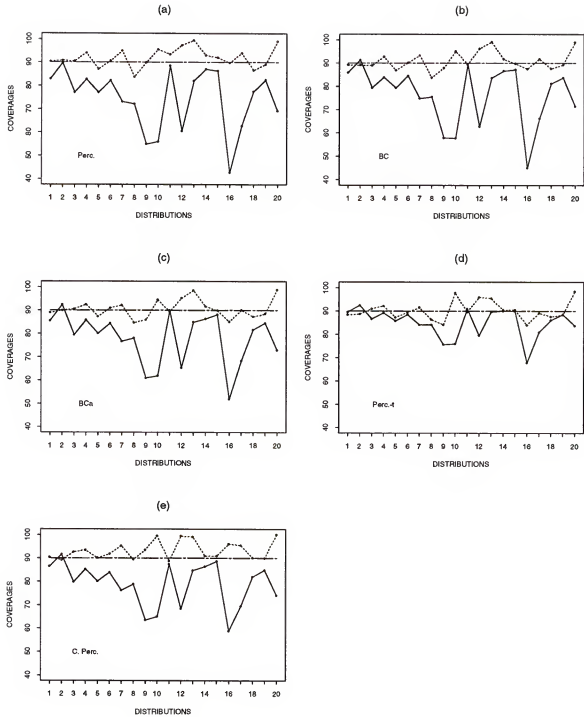


Figure 4.9. Coverage probabilities (in %) of a 90% confidence interval for σ^2 ($n=40$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

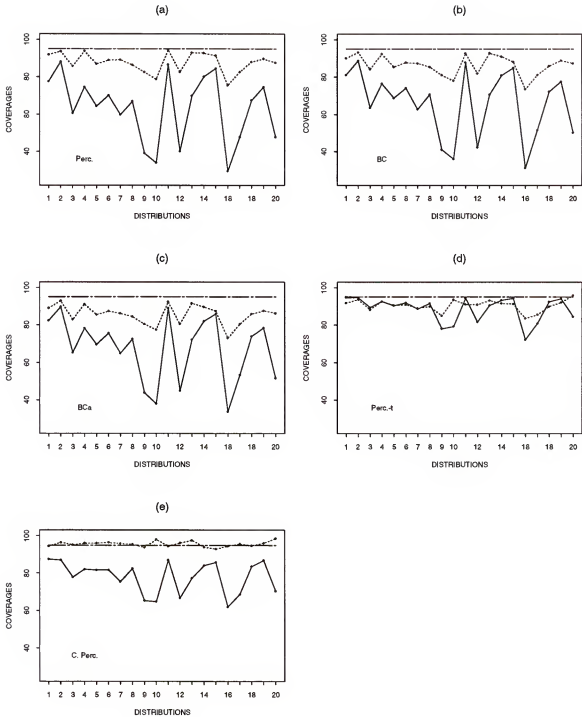


Figure 4.10. Coverage probabilities (in %) of a 95% confidence interval for σ^2 ($n=10$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

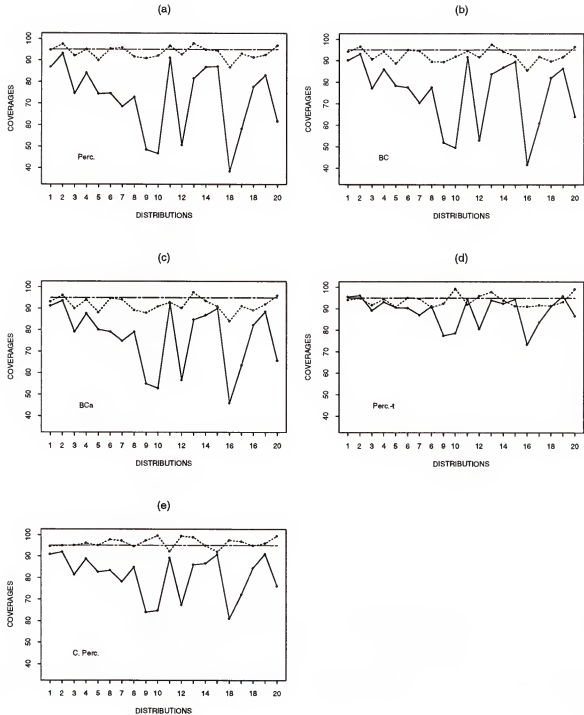


Figure 4.11. Coverage probabilities (in %) of a 95% confidence interval for σ^2 ($n=20$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

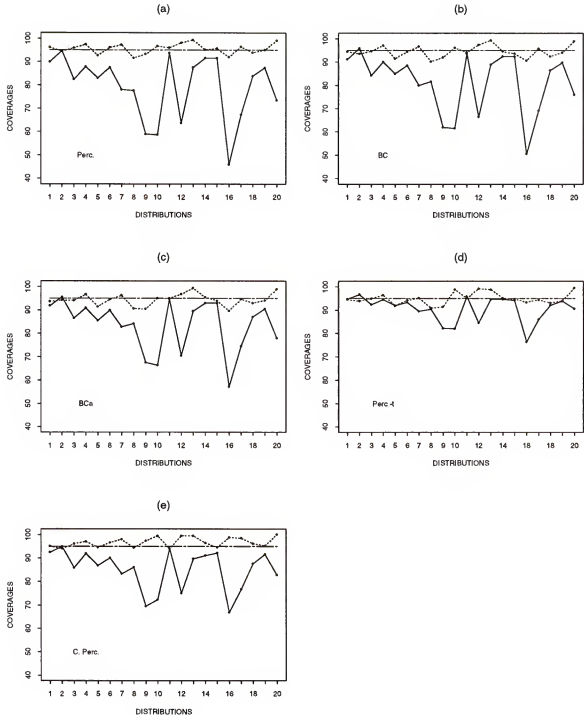


Figure 4.12. Coverage probabilities (in %) of a 95% confidence interval for σ^2 ($n=40$). (a) percentile method, (b) BC method, (c) BC_a method, (d) percentile-t method, and (e) centralized percentile method. Note: The values in the x-axis are the distributions in List 1 and List 2. The solid lines are for the ordinary methods and the dashed lines are for the modified method.

4.4 Trimmed Mean

When we consider the problem of estimating the center θ of symmetric distribution, we can use a mean or a median for the estimator of θ . The sample mean, however, is highly sensitive to heavy tails. The median can be used because of its insensitivity of outlying observations, but it may go too far in discarding observations. These considerations suggest a trimmed mean as a natural compromise between mean and median.

Let $(x_1, x_2, \dots, x_n)$ be a random sample from a symmetric distribution F and has a density f . The usual definition of the β -trimmed mean is given by

$$m(\beta) = \frac{1}{n - 2[n\beta]} \sum_{i=[n\beta]+1}^{n-[n\beta]} x_{(i)},$$

where $\beta \in [0, \frac{1}{2}]$, $[\cdot]$ denotes the greatest-integer function, and $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ are the order statistics. Bickel(1965) showed that under very general regularity conditions, $\sqrt{n} \cdot m(\beta)$ is asymptotically normal and the asymptotic variance is

$$\sigma^2(\beta) = \frac{1}{(1 - 2\beta)^2} \left\{ \int_{x_\beta}^{x_{1-\beta}} x^2 f(x) dx + 2\beta x_\beta^2 \right\},$$

where $x_\beta = F^{-1}(\beta)$ and $x_{1-\beta} = F^{-1}(1 - \beta)$. The estimate of $\sigma^2(\beta)$ is obtained by replacing the terms in the formula above by their estimates, i.e.,

$$s^2(\beta) = \frac{1}{(1 - 2\beta)^2} \left\{ \frac{1}{n} \sum_{i=[n\beta]+1}^{n-[n\beta]} (x_{(i)} - m(\beta))^2 + \beta(x_{[n\beta]+1} - m(\beta))^2 + \beta(x_{[n-n\beta]+1} - m(\beta))^2 \right\}.$$

To select the best β in $[0, \frac{1}{2}]$, Jaeckel(1971) proposed a method to use the trimmed mean whose asymptotic variance is minimum. He assumed a range of permissible

values of β , say $\beta_0 \leq \beta \leq \beta_1$, was fixed in advance. Then he found the minimum value of $s^2(\beta)$ for all β in this range such that $n\beta$ is an integer. He called the trimmed mean with this smallest asymptotic variance as the optimal trimmed mean. He recommends the range $[0, 0.25]$ for small sample. The trimmed mean and its estimate of $\sigma^2(\beta)$ based on the interval $[0, 0.25]$ will be denoted by $m_J(\beta)$ and $s_J^2(\beta)$, respectively. Then, the standard asymptotic confidence interval based on $m_J(\beta)$ is

$$\left(m_J(\beta) - z_{1-\alpha} \cdot s_J(\beta)/\sqrt{n}, m_J(\beta) - z_{\alpha} \cdot s_J(\beta)/\sqrt{n} \right). \quad (4.1)$$

Four different confidence intervals based on $m_J(\beta)$ were considered: the standard asymptotic interval (4.1), the centralized percentile interval (2.9), the percentile interval (2.7), and the percentile-t interval (2.14). The percentile-t interval is (2.14) with $s_J(\beta)$ replacing s .

We investigated the performances of these confidence intervals by a simulation study for $n=10, 20$ and $(1 - 2\alpha) = 0.90$. Since $m_J(\beta)$ assumes the symmetric distribution, we selected six symmetric distributions from List 1. Then, we simulated 800 samples of size 10 and 20 from the normal, uniform, mixed normal I, mixed normal II, double exponential, and t_2 distributions. The bootstrap confidence intervals were computed from 1000(=B) bootstrap samples.

Table 4.19 and 4.20 show the results for $n = 10$ and 20, respectively. We consider the standard normal distribution as a preferred distribution. The estimated coverage probabilities for the modified intervals are in parentheses. The correlation coefficients between the coverage discrepancies and ESKSD for the 6 distributions are given in the last rows of Tables 4.19 and 4.20.

It can be seen that the bootstrap intervals have better performances than the standard asymptotic interval based on $m_J(\beta)$. Also, we see that the modified method is a little better than the ordinary method. However, the gains in the coverage by using the modified method may not be significant in practice, especially for the case of the percentile-t method. The reason is the low correlation between the coverages

and ESKSD and due to the good performances of the ordinary confidence intervals, there was no need for modification. The important finding from this experiment is that the modified method was not harmful to the coverages, even in the case of a low correlation between the coverages and ESKSD. For the centralized percentile method with high correlation between the coverages and ESKSD, it can be seen that the modified interval had better performances than the ordinary centralized percentile interval based on $m_J(\beta)$.

Table 4.19. Coverage probabilities (in %) based on Jaeckel's method for the trimmed mean: $n = 10$, $1 - 2\alpha = 0.90$

Distribution	Stand.	C. Perc.	Perc.	Perc.-t	ESKSD
1. N(0, 1)	77.5	84.1(89.4)	87.6(91.3)	91.0(89.7)	0.183
2. U(-1, 1)	80.1	83.7(84.9)	86.9(89.9)	93.3(91.4)	0.190
3. MN I	77.5	89.0(90.1)	89.5(90.4)	91.0(90.8)	0.201
4. MN II	73.8	85.0(91.5)	85.4(90.5)	89.2(90.3)	0.208
5. D. Exp.	75.2	87.3(90.6)	86.6(90.9)	89.7(90.6)	0.190
6. t_2	78.4	89.4(88.2)	89.0(88.9)	90.0(89.4)	0.227
bias	12.9	3.6(1.6)	2.5(0.7)	0.9(0.7)	—
correlation	—	0.913	0.656	-0.131	—

Table 4.20. Coverage probabilities (in %) based on Jaeckel's method for the trimmed mean: $n = 20$, $1 - 2\alpha = 0.90$

Distribution	Stand.	C. Perc.	Perc.	Perc.-t	ESKSD
1. $N(0, 1)$	83.1	87.5(88.1)	90.5(90.8)	90.6(90.0)	0.134
2. $U(-1, 1)$	83.5	89.9(91.1)	86.8(89.5)	93.9(92.9)	0.148
3. MN I	79.6	88.4(89.5)	89.2(91.0)	89.7(90.0)	0.165
4. MN II	80.4	88.9(88.7)	89.7(90.1)	90.5(88.9)	0.144
5. D. Exp.	80.8	89.6(91.9)	90.1(90.4)	89.7(89.2)	0.159
6. t_2	80.3	89.4(92.4)	88.0(92.0)	90.5(91.6)	0.197
bias	8.7	1.1(1.5)	1.2(0.8)	1.2(1.1)	—
correlation	—	0.419	-0.358	-0.412	—

4.5 Correlation Coefficient

In this section, we consider the confidence interval for correlation coefficient ρ . It is known that Fisher's transformation of the sample correlation coefficient, assuming the underlying population is bivariate normal, has approximately normally distributed with variance independent of ρ . Let

$$Z \equiv \frac{1}{2} \log \left(\frac{1 + \hat{\rho}}{1 - \hat{\rho}} \right).$$

Then, the standard confidence interval for ρ based on Z is

$$\left(\tanh^{-1}(Z - z_{1-\alpha}[n - 3]^{-1/2}), \tanh^{-1}(Z - z_{\alpha}[n - 3]^{-1/2}) \right). \quad (4.2)$$

If the underlying population is not the bivariate normal, the bootstrap can be used for constructing confidence interval for ρ .

To construct the BC_a interval, we need to calculate the acceleration constant a , which is a function of U_i defined in (2.22). Since the U_i cannot be calculated exactly, the approximation as described in Section 2.2 is used.

The variance of $\hat{\rho}$ is given by

$$\sigma^2(\hat{\rho}) = \frac{\rho^2}{n} \left\{ \frac{\mu_{22}}{\mu_{02}^2} + \frac{1}{4} \left(\frac{\mu_{40}}{\mu_{20}^2} + \frac{\mu_{04}}{\mu_{02}^2} + \frac{2\mu_{22}}{\mu_{20}\mu_{02}} \right) - \left(\frac{\mu_{31}}{\mu_{11}\mu_{20}} + \frac{\mu_{13}}{\mu_{11}\mu_{02}} \right) \right\},$$

where $\{\mu_{ij}\}$ are the bivariate moments. The estimate $s^2(\hat{\rho})$ of $\sigma^2(\hat{\rho})$ is obtained by replacing the terms in the formula above by their estimates.

Five different confidence intervals will be considered: the standard interval (4.2), the percentile interval (2.7), the BC interval (2.15), the BC_a interval (2.20), and the percentile-t interval (2.14). The percentile-t interval is (2.14) with $s(\hat{\rho})$ replacing s .

We investigated the performance of these confidence intervals by a simulation study for $n = 20$. Let $\{(x_1, y_1), \dots, (x_n, y_n)\}$ be the sample drawn from a bivariate distribution. Two bivariate normal distributions with zero means and $\rho = 0$ or 0.5 are considered and denoted BN I and BN II. Also, the sample $\{(x_1, y_1), \dots, (x_n, y_n)\}$ is drawn such that

$$X = \gamma_1 + \gamma_2 Y + W,$$

where Y is a distribution of List 1, $W \sim N(0, 1)$, and W and Y are independent. Two values, $(0, 1)$ and $(0, 2)$, of (γ_1, γ_2) are used.

Table 4.21 shows the result for $(\gamma_1, \gamma_2) = (0, 1)$. The tabulated values are the coverages probabilities of a two-sided 90% confidence interval for the correlation coefficient based on 800 simulated samples from the 11 distributions. The values given in parentheses are the corresponding coverage probabilities for the modified confidence intervals. The first column of Table 4.21 gives the distributions of Y , except BN I and BN II. The sample size was 20 and the bootstrap replicates $B=1000$. The bivariate normal distribution (BN I) was considered as the preferred distribution. Since we cannot find ESKSD for a bivariate distribution, the ESKSD between the empirical

distribution of X and the standard normal distribution was calculated. Also, we calculated the ESKSD between the empirical distribution of Y and the standard normal distribution. The average of two ESKSDs is used for ESKSD in bivariate case. The last column of Tables 4.21 gives the ESKSD between the listed distributions and the preferred distribution.

The results show that the ordinary bootstrap confidence intervals are better than the standard interval based on Z and the modified intervals are better than the ordinary intervals. The percentile-t interval can be recommended for small sample bootstrap confidence interval due to its simplicity and good coverage probability.

Since $(\gamma_1, \gamma_2) = (0, 1)$ is used in Table 4.21, the population correlation coefficients are different. Table 4.22 shows the result for $\rho = 0.5$. In this case, we use $(\gamma_1, \gamma_2) = (0, \gamma_{2j})$, where γ_{2j} depends on the distribution j . The results also show that the modified intervals are better than the ordinary intervals.

Also, we considered another two cases: $(\gamma_1, \gamma_2) = (0, 2)$ and $\rho = 0.9$. The results are very similar to Table 4.21 and 4.22.

Table 4.21. Coverage probabilities (in %) of the bootstrap interval for the correlation coefficient: $\gamma_1 = 0$, $\gamma_2 = 1$; $n = 20$, $1 - 2\alpha = 0.90$

Distribution	ρ	Stand.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. BN I	.0	89.7	87.4(87.7)	88.9(88.5)	88.2(88.4)	92.5(91.4)	0.134
2. BN II	.5	89.1	86.8(88.6)	87.9(88.6)	87.4(89.5)	92.3(91.3)	0.135
3. U(-1,1)	.5	91.3	85.3(87.5)	87.7(88.0)	87.5(88.7)	92.3(90.4)	0.141
4. MN II	.935	87.4	85.3(88.6)	86.3(87.6)	86.5(87.9)	89.0(88.1)	0.143
5. t_4	.817	83.7	86.3(87.6)	85.5(86.0)	85.8(86.0)	87.7(86.5)	0.149
6. D. Exp.	.817	87.6	86.4(91.0)	87.0(89.7)	86.9(90.1)	88.1(92.0)	0.149
7. MN I	.866	80.9	84.2(89.5)	83.0(87.4)	83.1(87.7)	85.1(88.5)	0.155
8. χ_4^2	.943	82.1	84.7(93.1)	84.2(91.1)	83.9(91.8)	85.8(90.6)	0.170
9. Exp.	.707	82.7	84.1(92.5)	85.6(91.1)	84.9(90.4)	86.8(91.4)	0.173
10. LogN	.908	57.0	71.2(89.0)	69.1(87.4)	69.9(87.4)	74.6(86.4)	0.217
11. Weib.	.976	40.6	57.2(95.6)	56.0(95.6)	55.9(95.6)	74.3(96.5)	0.274
bias		11.0	8.3(2.2)	8.1(2.3)	8.2(2.0)	5.1(2.2)	—
correlation			-0.974	-0.977	-0.980	-0.933	—

Table 4.22. Coverage probabilities (in %) of the bootstrap interval for the correlation coefficient: $\gamma_1 = 0$ $\rho = 0.5$; $n = 20$, $1 - 2\alpha = 0.90$

Distribution	ρ	Stand.	Perc.	BC	BC_a	Perc.-t	ESKSD
1. BN I	.0	89.7	87.4(89.9)	88.9(89.0)	88.2(89.4)	92.5(89.1)	0.134
2. BN II	.5	89.1	86.8(90.3)	87.9(89.7)	87.4(89.6)	92.3(89.6)	0.135
3. U(-1,1)	.5	91.6	85.3(88.0)	87.7(87.7)	87.5(89.1)	92.3(88.1)	0.141
4. MN II	.5	89.0	86.5(90.3)	86.9(89.9)	87.4(90.4)	91.9(90.3)	0.140
5. t_4	.5	87.9	85.6(89.1)	87.6(88.6)	87.4(89.1)	89.0(88.1)	0.144
6. D. Exp.	.5	88.1	87.4(90.8)	88.1(89.2)	88.2(89.5)	92.0(89.5)	0.146
7. MN I	.5	86.9	84.4(91.5)	85.9(89.9)	85.4(90.3)	88.1(89.1)	0.150
8. χ_4^2	.5	88.1	86.9(91.4)	88.0(91.8)	87.7(92.0)	90.9(91.3)	0.155
9. Exp.	.5	86.4	84.5(91.9)	86.1(92.0)	86.1(91.5)	89.0(91.6)	0.169
10. LogN	.5	77.5	82.2(93.8)	81.6(92.4)	82.4(92.4)	82.2(90.9)	0.193
11. Weib.	.5	72.2	74.9(96.9)	74.0(96.7)	74.0(96.9)	78.8(94.7)	0.222
bias		4.3	5.3(1.8)	4.3(0.8)	4.4(1.5)	3.2(0.9)	—
correlation			-0.924	-0.947	-0.932	-0.957	—

4.6 Tests of Separate Families of Hypothesis

In this section, we consider the problem of testing a hypothesis that a population has a specified distribution against the alternative that its distribution belongs to a separate family, separate in the sense that an arbitrary simple hypothesis in the null can not be obtained as a special case of simple hypotheses in alternative hypothesis.

Let $(x_1, \dots, x_n)$ be a random sample from a distribution with density $h(x)$. Let the hypothesis of the separate families be $H_0 : h(x) = f(x, \theta)$ versus $H_1 : h(x) = g(x, \omega)$, where (θ, ω) are unknown vectors of parameters. Let $\hat{\theta}$ and $\hat{\omega}$ be the maximum

likelihood estimates of θ and ω . Let

$$\begin{aligned} F &= \log f(x, \theta), \\ G &= \log g(x, \omega), \\ F_{\theta_i} &= \frac{\partial \log f(x, \theta)}{\partial \theta_i}, \end{aligned}$$

and

$$T_n = n^{-1} \sum \log \{g(x_i, \hat{\omega}) / f(x_i, \hat{\theta})\}.$$

Cox(1961,1962) shows that under H_0 and appropriate regularity conditions,

$$Z_n \equiv \sqrt{n}(T_n - E_{\hat{\theta}} T_n) \rightarrow N(0, \sigma^2(\theta)),$$

where

$$\sigma^2(\theta) = V_{\theta}(G - F) - \sum \frac{C_{\theta}^2(G - F, F_{\theta_i})}{V_{\theta}(F_{\theta_i})},$$

and $V_{\theta}(\cdot)$, $C_{\theta}(\cdot, \cdot)$ denote variance and covariance under H_0 . Then the null hypothesis H_0 should be rejected if $Z_n > z_{1-\alpha} \cdot \sigma(\hat{\theta})$.

As an alternative solution, Epps et al.(1982) presented a test based on the difference between the sample moment generating function and its theoretical counterpart under H_0 .

A test based on the parametric bootstrap is proposed by Williams(1970). For a sufficiently large integer B , the bootstrap sample $\{(x_{1b}^*, \dots, x_{nb}^*), b = 1, \dots, B\}$ are drawn from the population with density $f(x, \hat{\theta})$, and the maximum likelihood estimates $(\hat{\theta}_b^*, \hat{\omega}_b^*)$ and $T_{nb}^* = n^{-1} \sum \log \{g(x_{ib}^*, \hat{\omega}_b^*) / f(x_{ib}^*, \hat{\theta}_b^*)\}$ are computed. Let $\hat{G}(t)$ be the empirical cdf of T_{nb}^* . Then the hypothesis H_0 is rejected at level α if $T_n > \hat{G}^{-1}(1 - \alpha)$. So, his test is actually the parametric percentile method based on the bootstrap cdf of T_{nb}^* .

Loh(1985) proposed a new test to improve Williams' test. For each θ, α , and n , define the critical value $c(\theta, \alpha, n)$ such that

$$P_{\theta}(T_n > c(\theta, \alpha, n)) = \alpha.$$

Suppose the distribution of T_n is continuous for all θ and $\sup_{\theta} P_{\theta}\{T_n > c_m(\alpha, n)\} = \alpha$, where $c_m(\alpha, n) = \sup_{\theta} c(\theta, \alpha, n)$. Since the critical value $c(\hat{\theta}, \alpha, n)$ of Williams' test is less than or equal to $c_m(\alpha, n)$, we have

$$\sup_{\theta} P_{\theta}(T_n > c(\hat{\theta}, \alpha, n)) \geq \alpha.$$

If we use a critical value for T_n that is greater than $c(\hat{\theta}, \alpha, n)$, the bias in the size of Williams' test can be reduced (Loh, 1985).

For another application of our modified method, we considered the well-known testing problem of $H_0 : h(x)$ = the lognormal density function versus $H_1 : h(x)$ = the exponential density function (Cox, 1962; Epps et al., 1982; Loh, 1985). Specifically,

$$H_0 : h(x) = f(x, \mu, \sigma)$$

$$H_1 : h(x) = g(x, b),$$

where

$$f(x, \mu, \sigma) = \{x\sigma(2\pi)^{1/2}\}^{-1} \exp\{-(\log x - \mu)^2/(2\sigma^2)\} \quad (4.3)$$

and

$$g(x, b) = b^{-1} \exp(-x/b).$$

From Cox(1962), the nominal level α rejection region is given by

$$\hat{\mu} - \log \bar{x} + \hat{\sigma}^2/2 > n^{-1/2} z_{1-\alpha} \{\exp(\hat{\sigma}^2) - 1 - \hat{\sigma}^2 - \hat{\sigma}^4/2\}^{1/2},$$

where $\hat{\sigma}^2 = n^{-1} \sum (\log x_i - \hat{\mu})^2$ and $\hat{\mu} = n^{-1} \sum \log x_i$. Also,

$$T_n = \log \hat{\sigma} + \hat{\mu} - \log \bar{x} + [\log(2\pi) - 1]/2.$$

Three different tests based on T_n will be considered: Cox's test, Williams' test and a test based on the percentile-t method.

We investigated the performance of these tests by a simulation study for $n=20$ and $\alpha = 0.05$. Since the null hypothesis assumes a lognormal population, we selected 11 lognormal distributions with different value of σ in (4.3).

We simulated 10,000 samples of size 10 from lognormal distributions with $\sigma = (1.0, (0.2), 3.0)$ and $\mu = 0$, the notation $(1.0, (0.2), 3.0)$ refers to all the values between 1.0 and 3.0 at 0.2 intervals. The reason to use 10,000 replicates, was to compared with Loh(1985)'s results. To find the bootstrap cdf of T_n , 1000(=B) bootstrap samples were used. Table 4.23 shows the result from the simulation. Since the best coverage comes around $\sigma = 1.4$, the lognormal distribution with $\sigma = 1.4$ is considered as a preferred distribution.

The Cox's test is quite unsatisfactory. This result is compatible with a similar simulation result in Loh(1985). The test based on the ordinary percentile-t method has good performance throughout all range of σ . We see that the modified Williams' test is much better than the ordinary Williams' test, especially when $\sigma \geq 2$.

Several columns in Table 4.24 are quoted from Loh(1985) and Epps et al.(1982) to compare our results. Comparing the results for $\sigma = 1.0$ and 2.0 in Table 4.23 with the ones in Table 4.24, we can see that the test based on the modified percentile-t method has better performance than Epps or Loh's tests.

Table 4.23. Probability (Type I error) for H_0 : Lognormal vs. H_1 : Exponential ($\alpha = .05, n = 20$)

σ	Cox	Williams	Perc.-t	ESKSD
1.0	0.002	0.044(0.055)	0.017(0.027)	0.173
1.2	0.000	0.064(0.073)	0.027(0.043)	0.149
1.4	0.000	0.053(0.071)	0.030(0.051)	0.142
1.6	0.000	0.047(0.054)	0.031(0.050)	0.149
1.8	0.000	0.035(0.040)	0.029(0.053)	0.162
2.0	0.000	0.023(0.035)	0.030(0.051)	0.178
2.2	0.000	0.016(0.031)	0.034(0.051)	0.194
2.4	0.000	0.012(0.027)	0.032(0.053)	0.210
2.6	0.000	0.010(0.028)	0.033(0.052)	0.225
2.8	0.000	0.008(0.025)	0.032(0.049)	0.239
3.0	0.000	0.007(0.028)	0.032(0.048)	0.251
bias	-	0.024(0.017)	0.020(0.004)	-
Corr.	-	0.910	-0.386	-

Table 4.24. Probability (Type I error) for H_0 : Lognormal vs. H_1 : Exponential ($\alpha = .05, n = 20$)

σ	Cox	Epps	Williams	Loh
1.0	0.002	0.066	0.049	0.0206
1.414	0.0001	0.066	0.061	0.0497
2.0	0.0001	0.066	0.024	0.0127

Source: Loh(1985) and Epps et al.(1982)

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH PROBLEMS

The bootstrap method was introduced as a resampling method to estimate the standard error of an estimate $\hat{\theta}$ for a parameter of interest θ . The basic idea, however, can be applied to a wide range of statistical problems. It is based on the idea that the available sample is the best possible representation of the distribution where the sample came from.

Confidence intervals constructed by the bootstrap method as suggested in the statistical literature have been largely for the purposes of defining refined bootstrap confidence intervals and of obtaining high-order asymptotic properties. However, confidence intervals constructed by the bootstrap method are sometimes unsatisfactory for small samples.

To improve the coverage of bootstrap confidence interval, the concept of preferred distribution is developed. The Kolmogorov-Smirnov distances between the data empirical distributions and the preferred distribution is then used to make the confidence interval more accurate.

The modified method produced improvement in coverage accuracy of order n^{-1} in two-sided intervals for the mean.

The modified confidence intervals have been applied to the cases of the mean, the variance, the trimmed mean, the correlation coefficient, and the testing problem of separate families of distributions in Chapter 4. Through a simulation study, it was found that the bootstrap method may treat the sample as if it came from a

normal distribution when θ is the mean, and from a uniform distribution when it is the variance.

Also, it has been found that the performance of the bootstrap confidence interval is highly correlated with the Kolmogorov-Smirnov distance between the sample and the preferred distribution, especially in the mean and variance cases. Moreover, the modified confidence intervals are shown to be better than the existing bootstrap confidence intervals considered. The idea of finding a preferred distribution for bootstrap method will be useful in other small sample studies.

A preferred procedure, as argued Hall(1986), Beran(1988) and Efron(1992), is to bootstrap an pivotal statistic such as a t -like statistic (2.11) in Section 2.2., since the ordinary percentile- t interval outperformed the other ordinary intervals in simulation studies.

It was also found that the modified percentile- t interval is better than the ordinary percentile- t interval. For the mean and variance cases, the other modified intervals, namely, the centralized percentile interval, the percentile interval, the BC interval, and the BC_a interval are also shown to be better than the ordinary percentile- t interval. Moreover, the other modified intervals have the similar coverage probability error as the modified percentile- t interval, except for the variance case with $n = 10$.

The modified method seems to be a promising approach for constructing confidence intervals, and more work needs to be done in utilizing the ideas of preferred distribution and the Kolmogorov-Smirnov distance. Application of the modified method to construct confidence intervals for other parameters of interest will be considered in the future. In this study, only the Kolmogorov-Smirnov distance was considered as a measure between the empirical cumulative distribution and the preferred distribution. Other distances such as the von Mises distance may also provide confidence interval improvement information. It is also an interesting research topic for the future.

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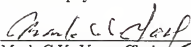
BIOGRAPHICAL SKETCH

Taewoon Kang was born on April 3, 1959, in Seoul, Republic of Korea. He was awarded a Bachelor of Economics degree in statistics in 1983 followed by a Master of Economics degree in statistics in 1987, from Sung Kyun Kwan University, Seoul, Republic of Korea. He served part-time as an instructor in the Government Computer Center until 1988.

In addition to pursuing his Ph.D. in statistics from the University of Florida, he has served as a statistical analyst and consultant for the Center for Gerontological Studies, the Departments of Geography, Urban and Regional Planning, Sociology, and the Division of Biostatistics. He has been a member of the American Statistical Association since 1992.

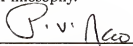
He is married and the father of a wonderful daughter. After graduation, he looks forward to research and teaching.

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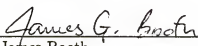
Mark C.K. Yang, Chairman
Professor of Statistics

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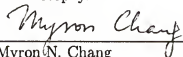
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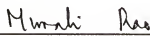
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This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 1995

Dean, Graduate School